

Lectures at the Summer Junior Workshop at UM madison

will discuss several extension problems in
SCV.

Lecture I : CR extension problems.

(1) Let $M \subset \mathbb{C}^n$ be a smooth submanifold.

$$\forall p \in M, CR_M(p) = \dim T_p^{1,0} M$$

$$T_p^{1,0} M = \{ L = \sum a_j \frac{\partial}{\partial z_j} \mid p, L \rightarrow \text{Tangent to } M \}$$

$$\text{at } p, \text{ i.e., } \forall f \in (C^\infty \otimes \mathbb{C}) \cap U \cap M, Lf(p) = 0 \}$$

$$f|_M = \text{const.}$$

(1)

M is called a CR manifold if $CR_M(p) = \text{const.}$

now, if $L_p \in T_p^{1,0} M$, then $\bar{L}_p \in T_p^{0,1} M$, also
tangent to M at p . Suppose M is CR. Then

$$T M = T^{1,0} M \oplus T^{0,1} M \oplus \mathcal{N}.$$

M is called of hypersurface type if $\dim_{\mathbb{C}} \frac{T M}{T^{1,0} M \oplus T^{0,1} M} = 1$

Assume M is orientable. Then $\mathcal{N} \cong \text{oriented}$. Then is

a global sect T s.t. $T|_p \in \mathcal{N}_p$. T can be chosen to be real

T is called a ~~complex~~ ^{contact} ~~normal~~ vector field of M at p

Θ a real 1-form Θ s.t.

$$\ker \Theta = T^{1,0}M \cap T^{0,1}M \text{ and } \langle T, \Theta \rangle = 1$$

$\Rightarrow$ called a contact 1-form of M .

Now, we can define a Levi-form w.r.t Θ .

$$\forall L_1, L_2 \in T^{1,0}M,$$

$$\lambda \quad \mathcal{L}(L_1, L_2)|_P = -i \langle \Theta, [L_1, L_2]|_P \rangle$$

$\mathcal{L}(L_1|_P, L_2|_P)$ depends only on the value of L_1, L_2 at P .

if $\lambda \geq 0$ (or $\lambda \leq 0$) at M , M is called pseudocconvex. ~~Otherwise~~ if $\lambda > 0$ (< 0), we call M strongly pseudocconvex. We always choose T s.t. $\lambda \geq 0$ or $\lambda > 0$ when M is pseudocconvex or strongly ψ -convex.

Let $M_1 \xrightarrow{F} M_2$ be a diffeomorphism for M_1, M_2 . Assume M_1 and M_2 are CR submanifolds of hypersurface type. we call F a CR diff. if

$$F_* (T^{1,0}_P M_1) = T^{1,0}_{F(P)} M_2 \quad \text{or } K \subset \mathbb{C} \text{ or } K = \mathbb{C}$$

(Note further Θ_1, Θ_2 , $\Theta_1 = K \Theta_2$ with $K \neq 0$. Then pseudo-convexity of M is indep't of the choice of Θ .

Let Θ_1 be a contact form of M_1 . Then $\Theta_2 = (F^{-1})^* \Theta_1$

$\Rightarrow$ also a contact form on M_2 and

$$\mathcal{O}_2(\overline{L_1}, \overline{L_2}) = \mathcal{O}_2(\overline{L_1}, \overline{L_2})$$

$$= -\mathcal{O}_2(\overline{F})^* \mathcal{O}_1([F \times L_1, F \times L_2])$$

$$= -i(F^{-1})^* \mathcal{O}_1([F \times L_1, L_2])$$

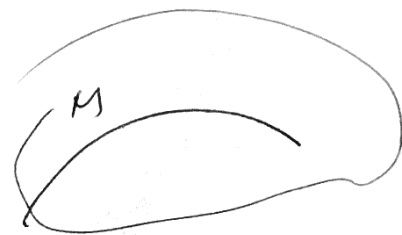
$$= -i \mathcal{O}_1([L_1, L_2]) = \lambda(L_1, L_2).$$

These ~~strongly~~ pseudocurves or strongly pseudocurves are invariant under CR transformations.

(2) Now, focus on hypersurfaces in $\mathbb{C}^n$:

Let $p \in \mathcal{O}(U)$, $M \subset U$ a smooth hypersurface. p is called a defining eq of M

if $\{p=0\} = M$ and $\nabla_{\mathbb{C}} p = (\frac{\partial p}{\partial z_1}, \dots, \frac{\partial p}{\partial z_n})|_M \neq 0$. ($\nabla p|_M \neq 0$).



$$\text{Define } \Omega_{\pm} = \{z \in U : \rho(z) \gtrless 0\}$$

(3)

M is oriented.

Assume $\rho_{z_n} \neq 0$ along M . Then

$$L_j = \frac{\partial}{\partial z_j} - \frac{\rho_{z_j}}{\rho_{z_n}} \frac{\partial}{\partial z_n} \rightarrow \text{Tangent to } M \text{ and}$$

is of (1,0)-type

$$(L_j(p)) = \rho_{z_j} - \frac{\rho_{z_j}}{\rho_{z_n}} \cdot \rho_{z_n} = 0$$

$\{L_1, \dots, L_{n-1}\}_p$ forms a basis of $T_p^{1,0} M$

$\{\overline{L}_1, \dots, \overline{L}_{n-1}\}_p$ forms a basis of $T_p^{0,1} M$.

$T^{1,0}M = \frac{1}{\pi} T_p^{1,0} \xrightarrow{\pi} M$ a smooth vector bundle over M of rank $n-1$ (complex)

A smooth section of $T^{1,0}M$ is called a CR vector field along M .

$\{L_1, \dots, L_{n-1}\}$ form a basis of all smooth CR vector fields along M .

Let $\mathcal{O} = \frac{1}{2i} (\partial\bar{\partial} - \bar{\partial}\partial)$. $T = 2 \operatorname{Im} \left(\sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} \frac{\partial}{\partial z_j} \right) / (\pi_2 P P)$

$T(p) = 0$. $T|_M$. $\langle \mathcal{O}, T \rangle = \frac{\sum |\frac{\partial f}{\partial \bar{z}_j}|^2}{|\pi_2 P|^2} = 1$

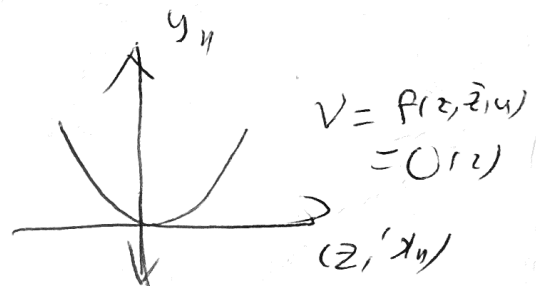
suppose $p=0$, $p \in M$: $p = -V + \mathcal{O}(z) = -\frac{R_n - \bar{z}_n}{2i} + \mathcal{O}(z)$

$T|_0 = 2 \operatorname{Im} \left. \frac{1}{2i} \frac{\partial}{\partial \bar{z}_n} \right|_0 = -\operatorname{Re} \left\{ \frac{\partial}{\partial x_n} + i \frac{\partial}{\partial y_n} \right\} / 2 = -\frac{1}{2} \frac{\partial}{\partial x_n} \Big|_0$

$J(T) = -\frac{1}{2} \frac{\partial}{\partial y_n} \Big|_0$

$J(T)$ is the outward normal direction of M at $p=0$.

$p=0$.



$\oplus TM = T^{1,0}M \oplus T^{0,1}M \oplus \mathbb{R}T$

$TM = H \oplus \mathbb{R}T$

$H = \operatorname{Re} T^{1,0}M = \operatorname{Re} T^{0,1}M$

H is called the contact bundle of M .

$$X = \sum \xi_j \frac{\partial}{\partial \bar{z}_j}, Y = \sum \eta_j \frac{\partial}{\partial \bar{z}_j} \in T^{(0,1)}M$$

$$\lambda(X, Y) = -i < 0, [X, \bar{Y}] >$$

$$\xi_j, \rho_j|_M, \eta_j, \rho_j|_M = 0.$$

when

$$\lambda(X, Y) = \sum \frac{\partial^2 \rho}{\partial \bar{z}_j \partial \bar{z}_\ell} \xi_j \bar{\eta}_\ell$$

$$\rho = -V + O(2).$$

$$\xi_n, \eta_n|_0 = 0$$

$$= \sum_{j, \ell=1}^{n-1} \frac{\partial^2 \rho}{\partial \bar{z}_j \partial \bar{z}_\ell} \xi_j \bar{\eta}_\ell$$

The eigenvals of $\left(\frac{\partial^2 \rho}{\partial \bar{z}_j \partial \bar{z}_\ell} \right)_{j, \ell=1}^{n-1} \rightarrow$ (called the

Wu-eigenvals. There are unique sets up to a scalar multiplication.

Normalization up to order 2:

Choose coordinates s.t. $\rho = -V + O(2)$ or $\rho =$

$$T_p M = \{v=0\}, T_0^{(0,1)} M = \{z_n=0\}.$$

$(\frac{1}{a})'$

with $\rho = -V + h(z, \bar{z})$. Then

$$h(z, \bar{z}) = \operatorname{Re} \sum a_{\alpha\beta} z_\alpha \bar{z}_\beta + \sum_{\alpha, \beta=1}^{n-1} c_{\alpha\beta} \bar{z}_\alpha \bar{z}_\beta +$$

$$\operatorname{Re} \sum b_\alpha u + du^2 + O(3).$$

$$z_n = u + iv.$$

$$\omega_{z_n}' = z_n - i \sum_{\alpha, \beta=1}^{n-1} a_{\alpha\beta} z_\alpha \bar{z}_\beta - i \sum b_\alpha z_n \bar{z}_\alpha$$

$-i d \bar{z}_n^2$ still write z_n for z_n' .

$$\operatorname{Im} V = \sum_{\alpha, \beta=1}^{n-1} c_{\alpha\beta} \bar{z}_\alpha \bar{z}_\beta + O(3)$$

$(\hat{x}_p)$ is a Hermitian mch. Then $\exists$

$(n-1) \times (n-1)$ invertible matrix P s.t. $P C P^{-1} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$

$z' = z P \rightarrow$ still we $z \neq z'$.

M:

$$I_m = - \sum_1^r |z_j|^2 + \sum_{r+1}^{n-1} |z_j|^2 + O(1)$$

$\min \{r\} \rightarrow$ called the signature of M at a .

~~note~~

CR functions: M CR submf in $\mathbb{C}^n$

f is called a CR func if V CR vector field

$\perp$, $\bar{Z}(f)|_M = 0$. (Ass $f \in C^1(M)$).

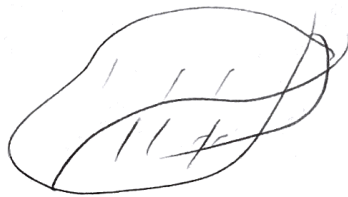
Ass M is a smooth hypersurf in $\mathbb{C}^n$. (Q)

Lemma: If $f \in C^2(M)$ and CR, then ~~Ass~~
 f extends to a C^2 smooth func $\tilde{f}$ in $\Omega(U)$

s.t. $\bar{\partial} \tilde{f} = O(\rho^2)$ aly M .

$$M = \{ \rho = 0 \}.$$

Pf: Let $L_a = \sum_{j=1}^n \left(\frac{\partial}{\partial z_j} \right)^2 / |\rho_z|^2$



Then $\{L_1, \dots, L_{n-1}, L_n\}$ forms a basis of $T^{1,0}\mathbb{C}^n|_M$

write its dual basis as $\{\omega_1, \dots, \omega_{n-1}, \omega_n\}$ of $(1,0)$ -forms

$$\omega_1 = \partial \rho \text{ as } \langle \partial \rho, L_j \rangle = \langle L_j, \rho = 0 \text{ to } j \leq n-1$$

$$\langle L_n, \bar{w}_n \rangle = 1$$

$$p \in \mathbb{C}R \Leftrightarrow \bar{\partial} f = \sum_{j=1}^n \bar{z}_j(f) \bar{w}_j = \bar{L}_n(f), \bar{w}_n = h_0 \bar{w}_n \text{ on } M_1$$

$$\bar{\partial}(f - h_0 p) = O(p) = p h_1 + O(p^2)$$

$$\text{since } \bar{\partial} \{ \bar{\partial}(f - h_0 p) \} = 0 \Rightarrow \bar{\partial} p \wedge h_1 + p \wedge \bar{\partial} h_1 = O(p) \text{ or}$$

$$\bar{\partial} p \wedge h_1 = O(p). \text{ Here}$$

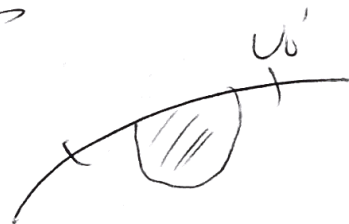
$$h_1 = h_2 \bar{\partial} p$$

$$\text{or } \bar{\partial} \{ f - h_0 p \} = p h_2 \bar{\partial} p + O(p^2)$$

$$\text{or } \bar{\partial} \{ f - h_0 p - \frac{1}{2} h_2 p^2 \} = O(p^2)$$

$$\text{Now } \tilde{p} = f - h_0 p - \frac{1}{2} h_2 p^2 \text{ does our job.}$$

Theorem 1 (Lay extension theorem) Let M be a smooth hypersurface in $\mathbb{C}^n$. Assume one of the Levi eigenvalues of M at $p \in M$ is non-zero. Then $\exists$ a neighborhood U' of p in M such that any CR function $f(z)$ in $M \cap U'$ extends to one side of M with $M \cap U'$ as part of its smooth boundary. This ~~one~~ domain is indep. of the choice of the CR functions.



pf: After a hol. change of coordinates, we assume that $p=0$ and M near p is defined by

$$\rho = -U + |z|^2 + \sum_{j=1}^{n-1} (g_j |z_j|^2 + O(|z_j|^3)), \quad g_j \in \mathbb{R}$$

$$\rho(z_1, 0) = |z_1|^2 + O(|z_1|^3).$$

choose $\delta > 0$, $0 < \varepsilon < \delta$ s.t.

$$z' = (z_2, \dots, z_n)$$

$$\frac{\partial^2 \rho}{\partial z_1 \partial \bar{z}_1} > 0 \quad \text{in} \quad \Omega = \{z \in \mathbb{C}^n \mid |z_1| < \delta, |z_1| + |z'| < \varepsilon\}$$

$$\rho > 0 \quad \text{in} \quad \partial\Omega \cap \{|z_1| \leq \delta, |z'| < \varepsilon\}$$

The set of $z_1 \in \mathbb{C}$ with

$$\rho(z_1, z') > 0 \rightarrow \text{connects for } |z'| < \varepsilon.$$

extend f to a function $\tilde{f}$ in Ω

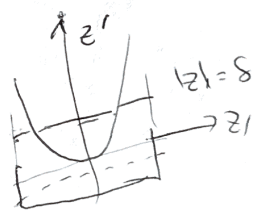
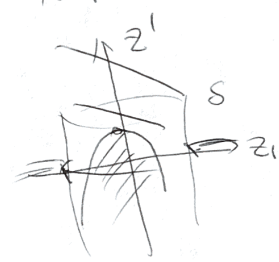
$$\text{s.t. } \bar{\partial} \tilde{f} = O(\rho^2) \text{ along } M.$$

$$\text{define } g = \begin{cases} 0 & \text{in } \Omega^+ \\ \bar{\partial} \tilde{f} & \text{in } \Omega^- \end{cases}$$

$$\text{then } g \in C_{(0,1)}^1(\Omega), \quad g = \varepsilon g_1 dz'$$

define

$$G(z_1, z') = \frac{1}{\partial \bar{\partial} z_1} \int_{\Omega} \frac{g(\xi, z')}{\xi - z_1} d\xi \wedge d\bar{\xi}. \quad |g_1|$$



$$122 \quad \frac{\partial g}{\partial \bar{z}_j} = \frac{1}{2\pi i} \int_D \frac{\frac{\partial g_1}{\partial \bar{z}_j}(\zeta, z')}{\zeta - z_1} d\zeta_1 d\bar{\zeta}$$

$$= \frac{1}{2\pi i} \int_D \frac{\frac{\partial g_1}{\partial \bar{z}_j}(\zeta, z')}{\zeta - z_1} d\zeta_1 d\bar{\zeta} = g_j$$

$$\frac{\partial g}{\partial \bar{z}_1} = g_1 \quad \text{by the Cauchy formula, } (z_1 = (1, 0))$$

$$\text{Hence } \bar{\partial}g = g \quad g = 0 \text{ in } \{z \in D, |z| < 1\}$$

$$g \text{ hol for } P(z, z') > 0 \text{ for each fixed } z'$$

$$\text{Hence } g \equiv 0 \text{ for } P(z, z') > 0$$

$$\text{or } g \equiv 0 \text{ in } \Omega_+. \text{ Now.}$$

$$\text{on } \Omega_- \quad \bar{\partial}(-g + \tilde{f}) = 0. \text{ Hence } -g + \tilde{f} \text{ is hol in } \Omega_-.$$

$$\text{and } -g + \tilde{f}|_M = 0. \text{ Thus } F = \tilde{f} - g \text{ extends hol.}$$

$$F \text{ to } \Omega^-. D_{z'} = \{z_1 : \text{by a domain}$$

$$\text{Notice that } P(z_1, z') < \alpha \text{ is a domain}$$

$$\text{in } D' \text{ for each fixed } z'. \text{ and thus } \exists$$

$$\begin{array}{c} \text{hol} \\ \{P(z, z') < \alpha\} \\ \varphi_z(\zeta) \cdot \Delta \rightarrow \{P(z, z') < \alpha\} \\ \varphi_z(\tau) \in M, \varphi_z(\zeta) \in \Delta \\ \varphi_{z'} \rightarrow M. \end{array} \quad (12)$$

$$\text{and } P(z_1, z') \in D_{z'} \subset M. \quad \{D_{z'}\}_{z' \neq 0} \text{ is a family}$$

$$\text{of hol. cur. attached to } M. \quad \Omega^- = \{(z_1, z') : |z_1| < \delta, |z'| < \epsilon\}$$

Strategy for general CR extension

- (1) - Baouendi - Trevis approximation then
- (2) - method of Bishop equals and attached holomorphic disks.

then (BT approximation)

Let $M \subset \mathbb{C}^n$ be a real hypersurface

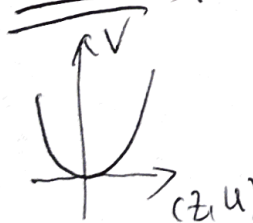
and $p \in M$. $\exists$ for any $w_1 \in P$, a neighborhood

p in M , $\exists$ w_2 another open subset of P such that

$\subset \{t - p \in w_2 \text{ and } \text{for any } \phi \in C^\infty(\mathbb{R})$

function f on w_1 , f can be ^{uniformly} approximated by hol. polynomials on w_1 . (1/6)'

proof Since M near $p=0$ is defined by



$$V = h(z, \bar{z}, u) = h(x, y, u)$$

$$h = O(|(x, y, u)|^2).$$

$$z = x + iy.$$

$$t = t(x, u)$$

Defn $H = \mathbb{R}^n \times \mathbb{R}^{n-1} \rightarrow \mathbb{R} \oplus \mathbb{R}^{n-1} \times \mathbb{R}^1 = \mathbb{R}^n$

$$(t, y) \mapsto (t, y, h(x, y, u))$$

Let $\mathcal{H}(x)$ be a Cⁿ function with

$$\text{supp } \mathcal{H} \subset U_1 \subset \mathbb{R}^n$$

$x \in I$ in a neighborhood of 0.

After modifying h by $x \mapsto \mathcal{H}(\frac{x}{\delta})h$ for

$0 < \delta < 1$, we can ensure that $|y| < \delta$.

$$|\frac{\partial \mathcal{H}}{\partial x}(x, y)| \leq \frac{1}{2} \text{ for } (x, y) \in \mathbb{R}^n \times \mathbb{R}^{n-1}.$$

$$(|A| = \sup_{|u|=1} |\mathcal{H}(u)|).$$

By the mean-value theorem.

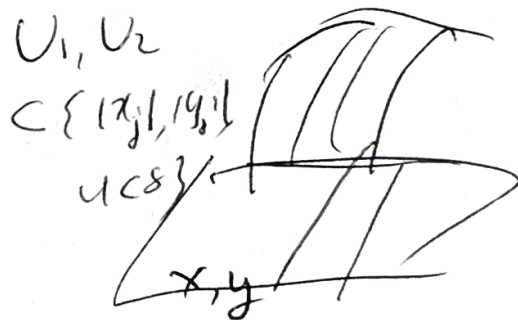
$$|\mathcal{H}(x_1, y) - \mathcal{H}(x_2, y)| \leq \frac{1}{2} |x_1 - x_2|$$

Define $M_y = \{(x, u) + iH : (x, y) \in U_1 \times U_2, \mathbb{C}\mathbb{R}^n \times \mathbb{R}^{n-1}\}$

$$\text{write } |z|^2 = s_1^2 + \dots + s_n^2$$

$$s \in \mathbb{C}^n, ds = ds_1 + \dots + ds_n.$$

M_y is totally real of dimension n in $\mathbb{C}^n$



$$\left(\frac{\partial}{\partial x} \left\{ \mathcal{H}\left(\frac{x}{\delta}\right)h \right\} = \frac{0(1)}{\delta} \cdot 0(\delta^2) + 0(1) \cdot 0(\delta) = 0(\delta) \right)$$

Lemma 1

Let $\chi \in C_c^\infty(U)$ with $\int \chi = 1$ in $U, \chi \geq 0$

$U_1' \subset U$, all $\chi = 0$ outside U . For every χ

to $\mathbb{R}^n$ by $\chi(Z_1) = \chi(Z)$

For $y \in U_2$

($Z = \epsilon t + i s$)

For $y \in M_y \cap \{U_1' \times U_2\}$, ~~define~~ then

$$f(y) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi^{n/2}} \epsilon^{-n} \int_{\varphi \in M_y} \chi(\varphi') f(\varphi') e^{-\frac{(\epsilon s - \varphi')^2}{\epsilon^2}} d\varphi'$$

then moreover, this limit is uniform for

$y \in U_2$ and $y \in M_y \cap \{U_1' \times U_2\}$. $ds' = ds_1 + \dots + ds_n$

Proof: Define $\varphi: \mathbb{R}^n \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ by

$$\varphi(t, y) = \epsilon t + i + H(\epsilon, y)$$

For $y \in U_2$, $t' \mapsto \varphi(t', y)$ for $t' \in U$

parameter M_y . Here, it suffices to show

Gauss kernel. ~~Gauss~~ Weierstrass approximation. $f \in C_c^\infty(\mathbb{R}^n)$
 $\varphi = \frac{1}{\pi^{n/2}} e^{-\frac{z^2}{\epsilon^2}}$
 $\int_{\mathbb{R}^n} \varphi \rightarrow 0$ $\lim_{\epsilon \rightarrow 0} \int \varphi_\epsilon \rightarrow f$ uniformly if $\varphi_\epsilon \in C_c^\infty(\mathbb{R}^n)$ $\int_{\mathbb{R}^n} \varphi = 1$
 $\varphi_\epsilon = \frac{1}{\epsilon^n} \varphi(\frac{x}{\epsilon})$

$$f(\xi(t, y)) \\ = \lim_{\varepsilon \rightarrow 0} \pi^{-n/2} \varepsilon^{-n} \int_{t \in \mathbb{R}^n} g(\chi(t')) f(\xi(t', y)) \cdot$$

$$e^{-\varepsilon^{-2} [\xi(t, y) - \xi(t', y)]^2} \det\left(\frac{\partial \xi}{\partial t'}(t', y)\right) dt'$$

When the limit is uniform in
 $(t, y) \in U_1' \times U_2$.

Replacing t' by $t - \varepsilon s$ (for t fixed), the above integral (P)

$$= \pi^{-n/2} \int_{s \in \mathbb{R}^n} \chi(t - \varepsilon s) f(\xi(t - \varepsilon s, y)) \cdot$$

$$e^{-\varepsilon^{-2} [\xi(t, y) - \xi(t - \varepsilon s, y)]^2} \det\left(\frac{\partial \xi}{\partial s}(t - \varepsilon s, y)\right) ds.$$

If $t \in U_1'$, $\chi(t - \varepsilon s) \rightarrow 1$ as $\varepsilon \rightarrow 0$. Moreover

$$\xi(t, y) - \xi(t - \varepsilon s, y) = \frac{\partial \xi}{\partial t}(t, y) \cdot (\varepsilon s) + O(\varepsilon^2).$$

Then

$$\frac{\partial \xi}{\partial t} = \begin{pmatrix} \frac{\partial \xi_1}{\partial t_1} - \frac{\partial \xi_n}{\partial t_n} \\ \frac{\partial \xi_4}{\partial t_4} - \frac{\partial \xi_n}{\partial t_n} \end{pmatrix}$$

$$\varepsilon^{-2} [\varphi(t, y) - \varphi(t - \varepsilon s, y)]^2$$

$$= \left[\varepsilon \frac{\partial \varphi}{\partial t}(t, y) \right]^2 + O(\varepsilon).$$

Hence, pointwise in s , the integrand converges, as

$\varepsilon \rightarrow 0$, to

$$f(\varphi(t, y)) e^{-\left(\varepsilon \frac{\partial \varphi}{\partial t}(t, y)\right)^2} \det\left(\frac{\partial \varphi}{\partial x}(t, y)\right).$$

To show the corresponding integrals converge, we must dominate the integrand by an integrable function.

$$\text{Show } \left| \varphi(t - \varepsilon s) f(\varphi(t - \varepsilon s, y)) \det\left(\frac{\partial \varphi}{\partial x}(t - \varepsilon s, y)\right) \right| \leq$$

C uniformly in s and ε .

$$\left| e^{-\varepsilon^{-2} [\varphi(t, y) - \varphi(t - \varepsilon s, y)]^2} \right|$$

$(5.6)'$

$$= e^{-\varepsilon^2 \operatorname{Re} \{ [\varphi(t, y) - \varphi(t - \varepsilon s, y)]^2 \}}$$

$$\text{Show } \varphi(t, y) = t + iH(t, y),$$

We obtain

$$\sum t_j^2 = \sum (t_j^1)^2 - 2\sum t_j^1 s_j + \sum s_j^2$$

$$\mathbb{R} \{ [\varphi(t, y) - \varphi(t - \varepsilon s, y)]^2$$

$$= \frac{\varepsilon^2 s^2}{t^2 - (t - \varepsilon s)^2} [H(t, y) - H(t - \varepsilon s, y)]^2$$

$$\geq \frac{1}{2} \varepsilon^4 |s|^2 \geq \frac{1}{2} \varepsilon^4 |s|^2$$

Here the exponential term is dominated by for all by $e^{-\frac{1}{2} |s|^2}$

where is an integrable fun of $s \in \mathbb{R}^n$

(6b)

By the Lebesgue dominated convergence theorem

as $\varepsilon \rightarrow 0$, the integrand converges to

$(f, g) \in U_1 \times U_2$ to

$$\pi^{-n/2} f(\varphi(t, y)) \int_{s \in \mathbb{R}^n} e^{-\frac{1}{2} \left[\frac{\partial \varphi}{\partial s}(t, y, s) \right]^2} \det \left(\frac{\partial^2 \varphi}{\partial s^2} \right) ds$$

which is but in $A = \frac{\partial^2 \varphi}{\partial s^2}$

$$\text{claim: } \pi^{-n/2} \int_{s \in \mathbb{R}^n} e^{-\frac{1}{2} (A \cdot s)^2} \det A ds = 1$$

$$A = \{ n \times n \text{ invertible } A : |I_w A| \in K(A) \}$$

$$A = \frac{\partial f}{\partial t} = Id + i \frac{\partial H}{\partial t}(t, y). \text{ Then}$$

$$|\operatorname{Im} A| \leq \frac{1}{2} |\operatorname{Re} A|.$$

A is connected ~~subset~~ open subset in $\mathbb{R}^{n \times n}$ containing the totally real submf $\operatorname{Im} A = 0$.

when $\operatorname{Im} A = 0$, $\det A > 0$

$$\pi^{-n/2} \int_{\mathbb{R}^n} e^{-[SA]^2} \det A \, ds = \pi^{-n/2} \int_{s \in \mathbb{R}^n} e^{-s^2} ds_1 \dots ds_n = 1.$$

$$I(A) = \pi^{-n/2} \int_{s \in \mathbb{R}^n} e^{-[AS]^2} \det A \, ds \text{ is hol in } A$$

$\in A$. Then $I(A) \equiv 1$.

Then the limit converges uniformly to $f(p(y))$.

Lemma 2: Suppose f is a CR function of class C^2 on U , with $0 \in U \subset \mathbb{C}^n$. Then $\exists U_2$ with $0 \in U_2 \subset U_1$

$$\text{s.t. } \forall \beta \in U_2, f(\beta) = \lim_{\varepsilon \rightarrow 0} \pi^{-n/2} \varepsilon^{-n} \int_{\beta \in M_0} \pi(\beta') f(\beta') e^{\frac{(\beta - \beta')^2}{\varepsilon^2}} d\beta'$$

Moreover, the limit is uniform in $\beta \in U_2$.

Def: We can extend f to a neighborhood of 0 ~~at~~ $\in \mathbb{C}^n$ s.t. $\partial f|_M \equiv 0$

(7.1)'

pf of lemma 2. For $\varphi \in \mathcal{Q}M$, we have

$$\varphi = t + iH(t, y) \text{ for a unique } t \in U, y \in U_2.$$

For $y \in U_2$, $y \neq 0$, let

$$\tilde{M}_y = \{ \varphi(t', \lambda y) = t' + iH(t', \lambda y) : t' \in U, (R^4, c\varphi) \leq 1 \}$$

$\tilde{M}_y$ is an (at)-dim sub of M and its boundary is the union of M_0, M_y and $\{ \varphi(t', \lambda y) : t' \in \partial U, 0 \leq \lambda \leq 1 \}$. (8b)

Since our integrand has compact support in U , the only contributing components of the boundary $\tilde{M}_y$ are M_0 and M_y . For $\varphi = \varphi(t, y)$, we obtain

$$f(\varphi) = \lim_{\varepsilon \rightarrow \infty} \pi^{-n/2} \varepsilon^{-n} \int_{\varphi' \in M_0} \mathcal{R}(\varphi') f(\varphi') e^{-\varepsilon^2 (P\varphi - P'\varphi)} d\varphi'$$

$$+ \lim_{\varepsilon \rightarrow \infty} \pi^{-n/2} \varepsilon^{-n} \int_{\varphi' \in \tilde{M}_y} \mathcal{R}(\varphi') f(\varphi') e^{-\varepsilon^2 (P\varphi - P'\varphi)} d\varphi'$$

Since $d = \partial + \bar{\partial}$, $d\bar{g} = d\bar{f}$, and $\bar{g}$ is a type (n,0)-form
 $\bar{\partial}f = 0$ on U_1 , we see that the second integral I_2

$$I_2 = \lim_{\varepsilon \rightarrow 0} \pi^{-n/2} \varepsilon^{-n} \int_{\substack{\bar{g}' \in \tilde{M}_Y \\ \bar{g}' = t + iH(t, y)}} \bar{\partial}_{\bar{g}'}(f(\bar{g}')) e^{-\frac{(\bar{g}-\bar{g}')^2}{\varepsilon^2}} d\bar{g}'$$

$\bar{g}' = t + iH(t, y)$. Need only show $\bar{g}' \in U_2$ $\bar{g} = t + iH(t, y)$

Since $\operatorname{Re} \{(\bar{g} - \bar{g}')^2\} = |t - t'|^2 - |H(t, y) - H(t', y)|^2$

we have

$$|H(t, y) - H(t', y)| \leq |H(t, y) - H(t', y)| +$$

$$|H(t', y) - H(t', y)| \leq \frac{1}{2} |t - t'| + C|y - y'|.$$

Then C is a constant depending only on $\|H\|_{C^1(\overline{U_1 \times U_2})}$.

$$\operatorname{Re} \{(\bar{g} - \bar{g}')^2\} \geq \frac{1}{2} |t - t'|^2 - 2C^2 |y|^2.$$

$\bar{\partial}_{\bar{g}'} \chi \equiv 0$ in a neighborhood of 0. If

$$\bar{g}' = t' + iH(t', y) \in \operatorname{supp} \bar{\partial} \chi,$$

$\delta \geq |t'| \geq \varepsilon_0 > 0$ for some $\varepsilon_0 > 0$. Note that

$$\bar{g} = t + iH(t, y) \rightarrow 0 \Leftrightarrow t \rightarrow 0, y \rightarrow 0.$$

Then, $\exists r, r_0 > 0$ s.t. if $\bar{g}' \in M_Y$, $|\bar{g}'| < r_2 < r_1$, ~~$|\bar{g}'| < r_1$~~

$$\bar{g}' \in \operatorname{supp} \bar{\partial} \chi \Rightarrow$$

$$\operatorname{Re} \{(\bar{g} - \bar{g}')^2\} \geq r_1.$$

$\bar{\partial} \chi$

 $M_{\lambda, y}$

$$\text{Hence } e^{-\varepsilon^2(p-p')^2} \leq e^{-\frac{r_1}{\varepsilon^2}}$$

$$\text{Hence } \left| \varepsilon^{-n} \int_{p' \in \tilde{M}_g} (\bar{\partial}_g' \chi) f(p') e^{\frac{(p-p')^2}{\varepsilon^2}} dp' \right| \leq C \varepsilon^{-n} e^{-\frac{r_1}{\varepsilon^2}} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

The proof of the lemma is done.

Proof of the Bochner-Tukey's approximation thm.

where $\omega_2 \subset \omega_1$ is sufficiently small, for $g \in M$, we have

$$\text{unique } f(g) = \int \lim_{\varepsilon \rightarrow 0} \pi^{-n/2} \varepsilon^{-n} \int_{M_0} \chi(g') f(g') e^{\frac{(g-g')^2}{\varepsilon^2}} dg'.$$

$$\text{Now } I_{\varepsilon}^g = \pi^{-n/2} \varepsilon^{-n} \int_{M_0} \chi(p') f(p') e^{\frac{(g-p')^2}{\varepsilon^2}} dp'$$

is an entire function. Hence the

proof is proved!

Note. We can further approximate I_{ε}^g by

a poly. we then conclude that $\exists \{P_n\}$ a

sequence of polys s.t. $P_n \rightarrow f$ uniformly on ω_2 .

Remark: If $f \in C^k(M)$ and M is C^{k+1} for $k \geq 2$,

then for any $p \in M$, $\exists p \in U_p \subset M$ s.t.

$\exists$ a sequence of hol. polys $\{P_j\}$ s.t.

$P_j \rightarrow f$ on $\bar{U}_p$ in $C^k(\bar{U}_p)$ -norm.

(See Baouendi-Simon-Rothschild).

$U_p \rightarrow$ then
no why
contin CR
fun ext approx

Cor. ^{then} Suppose M is a C^{k+1} -smooth real hypersurface.

$\forall p \in M$, let U_p be a ~~small~~ neighborhood of $p \in M$ s.t.

for any $f \in C^k(M)$ (~~continuous enough~~), $\exists \{P_j\}$, a

sequence of holomorphic polynomials, s.t. $P_j \rightarrow f$ in

C^k norm on U_p . Now suppose $\exists \{E_\alpha\}_{\alpha \in I}$ where

each E_α is a complex submanifold s.t. $\partial E_\alpha \subset U_p$ and

$\bigcup E_\alpha \supset \Omega$ s.t. Ω has a piecewise open side of M

near p as part of its smooth body. Then

if any $f \in C^k(M) \cap C^k(M)$, $\exists$ a hol. func $F \in \Omega$

s.t. $F \in C^k(\Omega \cup \omega^0)$ and $F|_{\omega^0} = f|_{\omega^0}$. (11)'

pf: let $P_j \rightarrow f$ uniformly on $\bar{U}_p$ in $C^k(\bar{U}_p)$ -norm.

By the max principle $\sup_{\Omega} |D_z^\alpha P_j - D_z^\alpha P|$

$$\leq \sup_{z \in U \cap \bar{E}_\alpha} |D_z^\alpha P_j - D_z^\alpha P_\ell| \leq \sup_{z \in D \cap \bar{E}_\alpha} |D_z^\alpha P_j - D_z^\alpha P_\ell|$$

$$\leq \sup_{\bar{U} \cap \bar{E}_\alpha} |D_z^\alpha P_j - D_z^\alpha P_\ell| \rightarrow 0 \text{ as } j, \ell \rightarrow \infty.$$

Here P_j converges in C^∞ locally to F , which does our job.

Or: Let M have at least one non-zero Levi eigenvalue at $p \in M$, $\exists$ ~~open set~~ and D_0 s.t.

∂D_0 has a piece $\partial U_p \cap p$ of M s.t. any CR function f over M (C^∞ smooth), $\exists$

$F \in \text{Hol}(D_0 \cap U_p) \cap C^\infty(\bar{D}_0 \cap U_p)$ s.t.

$$F|_{\partial U_p} = f.$$

(12)₆

Pf. Some $p \in M$ at new p is def'd by

$$p = -U + |z|^2 + O(|(z, w)|^3). \quad 0 < \varepsilon < \delta < 1$$

Let ε, δ be s.t. $z' = (z_1, \dots, z_n) \in U_\varepsilon$, $|z'| < \varepsilon$

$$|z_1| > \delta$$

s.t. $p(z_1, z') > 0$. Next for any let

$$\Omega = \{(z_1, z') : |z_1| < \delta, |z'| < \varepsilon : p(z_1, z') < 0\}.$$

We first choose $W_{p=0} \supset \{z \in M, |z| < \delta, |z'| < \varepsilon\}$

second for any $(z_1^0, z_1'^0) \in \Omega_-$.

$\rho(z_1^0, z_1'^0) < 0$. We then, we mentioned before

$\partial \bar{E}_{z_1^0} = \{z_1 \in \mathbb{C}^1: \rho(z_1, z_1'^0) < 0\}$ is compact

and $\partial \bar{E}_{z_1^0} \subset W_{p=0}$. By the Poincaré lemma

then, f extends to $F \in \text{Hol}(\Omega_-) \cap C^k(\Omega_- \cup \partial \bar{E}_{z_1^0})$,

$(|z| < \delta, |z'| < \varepsilon) \cup W_{p=0}^*$.

$W_{p=0}^* = \{z \in M: |z| < \delta, |z'| < \varepsilon\}$

(13) b.

Theorem of Treppeau:

Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface, $p \in M$.

Then the following statements are equivalent

- (1) $\forall B_p(\varepsilon)$ the union of all hol. disk attached to M ^{$\cap B_p(\varepsilon)$} ~~at~~ p :

$$\mathcal{H} = \bigcup_{\substack{\Delta \xrightarrow{\text{hol}} \mathbb{C}^n, \quad \varphi(\partial\Delta) \subset B_p(\varepsilon) \cap M \\ \text{cont. up to } \bar{\Delta} \text{ s.t. } \varphi(\bar{\Delta}) \subset \mathbb{C}^n}} \Delta$$

s.t. Δ contains a neighborhood on one side of M with
 a small piece of M at p as part of its smooth
 boundary

- (2): $\nexists$ a complex hyper $\bar{E} \ni p$ s.t. $\bar{E} \subset M$.
 or M is ~~non-min~~ minimal at p .

Combining with the BT approximation theorem, we see that
 Cor. any CR function of M extends to at least one side of
 M near p iff M is minimal at p .

A hol. disk $\varphi: \Delta \rightarrow \mathbb{C}^n$, $\varphi(\partial\Delta) \subset M$ is called
 a Bishop disk. The study of Bishop disks is
 of fundamental importance in many fields of math.

(14)6.

Theorem (Harvey-Lawson): Let $M \subset \mathbb{C}^n$ be a compact smooth strongly pseudoconvex CR submanifold of hypersurface type. ~~Assume that~~ Then after possibly a hol. re-embedding, if needed, M bounds a unique Stein space $\hat{M}$ with isolated ~~singularities~~ singularities, where $\hat{M}$ has M as its strongly pseudoconvex smooth boundary.

Proof: (i) Extend M to ~~the boundary of~~ a complex submanifold with M as part of its smooth boundary.

Let $p \in M$. After a hol. change of coordinate we assume that $p = 0$, $T_p^{(1,0)} M = \{z_{k+1} = \dots = z_n = 0\}$

$$T_p M = T_p^{(1,0)} M \oplus T_p^{(0,1)} M \oplus T_p^{(1,1)} M = \{z_{k+1} = 0, z_{k+2} = \dots = z_n = 0\}$$

Then M near $p = 0$ is defined by

$$z_{k+j} = h_j(z_1, \dots, z_k, \bar{z}_1, \dots, \bar{z}_k, z_{k+1}) \quad j=2, \dots, n-k$$

$$z_{k+1} = h_0(z_1, \dots, z_k, \bar{z}_1, \dots, \bar{z}_k, z_{k+1})$$

$$h_j, h_0 = O(2)$$

Define $\pi: \mathbb{C}^n \rightarrow \mathbb{C}^{k+1}$
 $z \mapsto (z_1, \dots, z_{k+1})$.

Let $U \ni 0$ be a small neighborhood of 0 in $\mathbb{C}^n$.

π is hol and $\pi|_{M \cap U_0}$ has a smooth
 inverse.

$$\psi: (z_1, \dots, z_k, z_{k+1}) \mapsto$$

$$(z_1, \dots, z_k, \gamma_{k+1}(z_1, \dots, z_k), \gamma_{k+2}(z_1, \dots, z_k), \dots, \gamma_n(z_1, \dots, z_k))$$

$$\text{from } M_0 = \{(z_1, \dots, z_{k+1}) : \gamma_{k+1} = h_0(z_1, \dots, z_k, z_{k+1})\}$$

$\pi|_{M \cap U_0}$ is CR and thus ψ is also CR.

(Fact: $M_1 \xrightarrow{f} M_2$, f is a smooth diffeo. + f is
 CR, then f^{-1} is CR).

M_0 is also strongly $\mathbb{C}$ -convex and M is assumed to
 be. By the Lewy extension theorem, ψ extends

a smooth hol. map, for D_p a certain $\mathbb{C}$ -convex side D_p^0

of M_0 . ψ^* is also a bihol. map if
 we shrink D_p^0 .

(can assume that D_p^0 is filled in by small

hol. discs attached to M_0 near P)

Now, we define $D_p = \mathcal{U}(D_p^0)$. Then D_p is a complex submanifold with point p as part of its smooth boundary.

By selecting D_p^0 carefully and assigning a Riemannian metric g on M , we have that $U_p^{*(1)}$ is a

convex open piece in (M, g) . Now, we choose enough points $\{p_1, \dots, p_m\}$ s.t. $\bigcup U_{p_j}^{*(1)} = M$.

We further choose $U_{p_j}^{*(2)} \subset U_{p_j}^{*(1)}$ s.t.

$U_{p_j}^{*(2)}$ is also convex in (M, g) and

$\bigcup U_{p_j}^{*(2)} = M$. and choose $\overline{D_{p_j}^{*(2)}} \subset \overline{U_{p_j}^{*(2)}}$ with

$$\partial D_{p_j}^{*(2)} \cap M = U_{p_j}^{*(2)}.$$

(3')_c.

Define $D = \bigcup_{j=1}^m D_{p_j}^{*(2)}$. Then $\partial D \supset M$.

We prove that D is a complex submanifold near M .

Let $0 < \varepsilon < 1$ be s.t. $\forall p \in \overline{D_{p_j}^{*(2)}} \cap U_{p_j}^{*(2)}$ then consider all hol. embedded disks attached to M near p with distance $< \varepsilon$ to p . Let $\mathcal{R}_p$ be its union. Then $\mathcal{R}_p$ is a submanifold of D_{p_j} defn. before at $\partial \mathcal{R}_p \supset$ an open of p in M .

we let $p, q \in D_{p_+}^{*(12)} \cap D_{p_-}^{*(12)}$. Take $\varepsilon < 1$ and

assume $\text{dist}(q, M) < \varepsilon$. Then there is a

hol disk φ_q in $D_{p_+}^* \cap D_{q_-}^*$ passing thru q .

and a small deformation of φ_q takes an open

neighborhood of q in $D_{p_+}^{*(12)} \cap D_{p_-}^{*(12)}$. Now, by
~~the uniqueness of complex submanifold~~ D is a complex

submanifold near q . Apparently $D \rightarrow \text{smoothly}$
along M . Here we proved that

$D \cap \{z : \text{dist}(z, M) < \varepsilon\}$ is a complex
submanifold with M as part of its smooth
boundary.

(4)'.
c.

(2) Now, suppose $M \subset D$ with D a ^{balanced} pseudocycle
down.

write $M_S = \{z \in D : \text{dist}(z, M) < \delta\}$. Then

M_S is a complex submanifold in $D \setminus K$ with

$K \subset D$, $K = \{z \in D : \text{dist}(z, M) \geq \delta\}$.

By the Hartogs extension theorem, we conclude that

M_S extends to a complex analytic var $\tilde{M}$.

Since the singular of $\tilde{M}$ is also a complex
analytic var of codim $\tilde{M}$ is smooth near M ,

the set of singularities has to be isolated. This

completes the proof in the case $M \subset \partial D$.

(5) i.e.

~~when in the general case, we need to embed~~

~~M_S into $\mathbb{P}^N \setminus \Delta(0,1)$ and $\setminus \Delta^N(0,1-\delta)$ and then~~

~~apply the Hartogs extension theorem.~~

Remark: we need only the following properties for M :

- (1) M 4-convex (2): $\forall \varphi \in M$ is minimal at each
 $p \in M$ by Treppe theorem.

General setting

- (3). For each $p \in M$, we claim $\exists$ a smooth CR function h_p s.t. $h_p(p) = 1$, $|h_p(q)| < 1$ for $q \neq p \in M$. By the very extension theorem, h_p extends holomorphically to M_S . The proof uses the technique from $\bar{\partial}$ -equation. We first find a CR (smooth) function g near p s.t. $g(p) = 0$, $\operatorname{Re} g(q) \simeq -\frac{1}{2}|g-p|^2$. Let π_p be a cutoff function near p and consider

$$\bar{\partial} u = \omega = \bar{\partial}(\pi_p \log g(p)). \text{ Then since } \omega \perp \mathcal{H}^1(M),$$

$$\text{we can find } u \in C^\infty(M) \text{ s.t. } \bar{\partial} u = \bar{\partial}(\pi_p \log g(p)).$$

$$\text{Let } v = \exp(-u + \pi_p \log g), \text{ then } v(p) = 1,$$

$$v \neq 0, q \neq p, \text{ and } \exists \delta > 0 \text{ s.t.}$$

$$v(M) \cap \{z \in \mathbb{C}^n : |z - 2\delta| < \delta\} = \emptyset.$$

$$\text{Now, let } \eta \text{ be a conformal map from } \{z \in \mathbb{C}^n : |z - \delta| > \delta\}$$

$$\text{to the unit disc which maps } 0 \text{ to } 1. \text{ Then}$$

$$\eta \circ v \text{ does our job.}$$

$$\text{Let } K_\varepsilon^0 = \{z \in M_S : \rho(z) > \varepsilon, 0 < \varepsilon < \rho\}$$

$$\text{Pick points } \{p_1, \dots, p_{n_0}\} \subset M, \text{ we can define}$$

$$\text{a hol. mapping}$$

$$F: M_S \rightarrow \mathbb{C}^N$$

We can define a list mapping

$$F: M_S \rightarrow \mathbb{C}^N$$

$$z \mapsto (z, t_{p_1}^{m_1}, \dots, t_{p_{N_0}}^{m_{N_0}})$$

with $c_0 \gg 1, m_1, \dots, m_{N_0} \gg 1$.

$$\text{s.t. } F(M_S) \subset \{ |w_j| < B = \frac{1}{c_0} \}$$

$F(K_S^0) \cap P(1-S', Y_L)$ is a closed complex submf of $\dim k$, where

$$P(1-S', Y_L) = \{ \frac{1}{c_0} < |w_j| < 1-S', \text{ for some } j \}$$

$$= P(1-S') \setminus P(Y_L)$$

Apply the Harkov then for complex analytic variety as before, we are done!

Topic two: ~~the~~ Carathéodory Rigidity Theorem.

$$\ell^2 = \{a = (a_1, a_2, \dots) : \|a\|_{\ell^2}^2 = \|a\|^2 = \sum_{j=1}^{\infty} |a_j|^2, \\ a \in \mathbb{C}\}$$

Hilbert space equipped with the standard ℓ^2 norm defined above.

Let M be a complex manifold of complex dimension n
A hol. map

$$F = (f_1, \dots, f_n, \dots) : M \rightarrow \ell^2$$

is called hol if

(a): each f_j is hol. over M

(b): $\sum_{j=1}^{\infty} |f_j|^2$ converges uniformly on compacta

Suppose $M = U \subset \mathbb{C}^n$. Then

$$H(z, \bar{w}) = \sum_{j=1}^{\infty} f_j(z) \overline{f_j(w)} \text{ converges uniformly on}$$

compacta in $U \times \text{conj}(U)$. $\text{conj}(U) = \{z : \bar{z} \in U\}$

(also $\sum_{j=1}^{\infty} \partial_z^\alpha f_j \cdot \overline{\partial_{\bar{z}}^\beta f_j}$) converges uniformly on

compacta over $U_p \times \text{Comp}(U)$. to

$$\partial_z^\alpha \partial_{\bar{z}}^\beta H(z, w).$$

$\sum (\partial_z^\alpha f_j)^2 \Rightarrow$ converges uniformly on compacta on U

(c) $\partial_{\bar{z}}^\alpha \bar{F} = (\partial_{\bar{z}}^\alpha f_1, \dots, \partial_{\bar{z}}^\alpha f_n, \dots)$ is also
a hol. map fr $U \rightarrow \mathbb{R}^2$.

now

$$F(z) = \sum \frac{\partial_z^\alpha F(p)}{\alpha!} (z-p)^\alpha$$

$\forall p \in U$. Converges uniformly on compacta in $\mathbb{R}^2$.

(2)

~~Let X_F~~

write X_F for the closure of the linear span
of $F(M)$ in $\mathbb{R}^2$.

~~Let $p \in M$~~ Let $p \in M$ and (q, z) near p with
 $z(p) = 0$.

Then ~~$\partial_z^\alpha F$~~ $F(z) = \sum \frac{\partial^\alpha F(0)}{\alpha!} z^\alpha$

claim. ~~the~~ ~~note that~~ $\partial^\alpha F(0) \in X_F$. Induct ^{for ev} $p_j = 0, 1, \dots$

$$\frac{F(0 + he_j) - F(0)}{h} \in X_F.$$

$$\partial_{z_j} F(0) = \lim_{h \rightarrow 0} \frac{F(0 + he_j) - F(0)}{h} \in X_F.$$

By an induct, we have $\partial^\alpha F(0) \in X_F \forall \alpha$.

Conversely, $F(z) \in \text{Span}\{\partial^\alpha F(0)\}$ for $z \in D$.

Next, $\partial^\alpha F(z) = \partial^\alpha F(0) + \partial^\alpha F(1)$

$$= \sum_{\beta} \frac{\partial^{\alpha+\beta} F(0)}{\beta!} z^\beta$$

$$\partial^\alpha F(z) = \sum \frac{\partial^{\alpha+\beta} F(0)}{\beta!} z^\beta \in X_F \cdot \text{Span}\{\partial^\alpha F(0)\}.$$

To apply the Taylor expansion at z_0 , we substitute

$$\forall z \in D. F(z) \in \text{Span}\{\partial^\alpha F(z_0)\} = X_F. \quad (3'')$$

Here Σ_F is the closure of the ^{span of} jets of F
at any given point.

claim $X_{hF} = X_F$. $\forall$ local.

define: $p^{\infty} = \mathbb{R}^2 \setminus \{0\} / \sim$.

$x, y \in p^{\infty}$, $x \sim y$ if $x = ky$.

$\neq 0$ for $(x_1, \dots, x_n) \in \mathbb{R}^2 \setminus \{0\}$

$[x_1, \dots, x_n]$ is called the homogeneous coordinates
of p^{∞} .

p^{∞} is called the infinitely close complex projective

space a copy of $\mathbb{R}$.

(4)''

A map $[F]: M \rightarrow p^{\infty}$ is called a helix

if $\forall p \in M, \exists U_p \subset M$ and a helix p_p ^{represent}

$F = (f_1(z), \dots, f_n(z), \dots)$ in $(f_1(z), \dots, f_n(z), \dots)$

$\Sigma: U_p \rightarrow \mathbb{R}^2 \setminus \{0\}$ is a helix.

only (1): $\forall z \in U_p, \exists f_j(z) \neq 0$

(2) Σf_j curve map on curve on U_p .

The generalised Fubini-Study metric, denoted by

ω_{FS} , of $\mathbb{P}^n$ with homogeneous coordinates

$(z_0, \dots, z_n)$ is given by

$$\omega_{FS} = \frac{i}{2\pi} \partial \bar{\partial} \log \sum_{j=0}^n |z_j|^2.$$

Let (M, ω) be a Kähler manifold with Kähler form ω .

~~$F = (f_0, \dots, f_n)$ be a holomorphic map~~

representations
~~map~~

F is a holomorphic map $M \rightarrow \mathbb{P}^n$.

$\forall p \in U$, $F|_U = (f_0, \dots, f_n)$ is a holomorphic map

if $\int_U \frac{i}{2\pi} \partial \bar{\partial} \log \sum_{j=0}^n |f_j|^2 = \omega|_U$

The F is called a holomorphic embedding of

(M, ω) into $(\mathbb{P}^n, \omega_{FS})$.

If F may not be one-to-one but has to be locally ~~locally~~ one-to-one.

projectivization

$[X]_F [X_F] = X_F$ is called a ~~linear~~ ^{projectivization} of X_F .

$(f)''$

Theorem (Calabi Rigidity theorem): Let M be a complex m.f. Let

$\mathbb{F}, \mathbb{G} = (M, \omega) \rightarrow \mathbb{P}^\infty$ be two hol. maps s.t.

$$f^*(\omega_{\mathbb{F}}) = g^*(\omega_{\mathbb{G}}). \text{ Then}$$

$\exists$ a linear isometry T from $X_{\mathbb{F}}$ to $X_{\mathbb{G}}$ s.t.

$$\mathbb{G} = [T] \circ \mathbb{F}$$

($X_{\mathbb{F}}$ is well-defined as $X_{h\mathbb{F}} = X_{\mathbb{F}}$).

Ex: $\overline{[\mathbb{F}]}$ may not be able to extend to a linear isometry from $\ell^2 \rightarrow \ell^2$

⑥''

$$\mathbb{F}: \Delta \rightarrow \mathbb{P}^\infty$$

$$z \mapsto [1, z, z^2, \dots, z^m, \dots]$$

$$\mathbb{G}: \Delta \rightarrow \mathbb{P}^\infty$$

$$z \mapsto [0, z, z^2, \dots, z^m, \dots]$$

$T: e_j \rightarrow e_{j+1}, \quad j=1, 2, \dots$ T does not extend to a linear isomorphism of ℓ^2 .

$$\mathbb{G} = [T] \circ \mathbb{F}$$

$U(z)$ be a small hol. chart on M with
 $z(U)$ a small ball centered at $0 \in \mathbb{C}^n$.

$$\mathbf{F} = \{F_j\}: F = (F_1, \dots, F_{n-1}), F = [F]$$

$$\mathbf{G} = [G]: G = (G_1, \dots, G_{n-1}), G = [G].$$

$$\|F(z)\|^2 = \sum |F_j|^2, \|G\|^2 = \sum |G_j|^2$$

And apply a hol. isomorphism of $\mathbb{C}^2$ w. Cauchy

$$\text{tht } F_1 = G_1 = 1, F_j(0) = 0,$$

$$G_1 = 1, G_j(0) = 0.$$

$$[F]^p(w_{s+1}) = [G]^p(w_{s+1}) \Leftrightarrow$$

$$\partial \bar{\partial} \log(\|F(z)\|^2) = \partial \bar{\partial} \log(\|G(z)\|^2). \quad (7)''$$

$$\log\left(1 + \sum_{j=2}^{\infty} |F_j|^2\right) - \log\left(1 + \sum_{j=2}^{\infty} |G_j|^2\right) = \text{Re } h(z)$$

$$h \text{ hol and } h(0) = 0$$

In the Taylor expansion the left, no problem
 to handle, but the left hand side.

$$R(z) = 0$$

How we get

$$\|F(z)\|^2 = \|G(z)\|^2 \quad z \in U$$

Complexifying, we get

$$\langle F(z), F(w) \rangle = \langle G(z), G(w) \rangle \quad z, w \in U.$$

Defn $X_F^0 = \text{span}\{\partial^\alpha F(w) = \text{circled out}, \alpha = \{\alpha_1, \alpha_2\} \in \mathbb{N}^2\}$

Then $X_F = \overline{X_F^0}$ in $\mathcal{O}^2$.

We can similarly define X_G^0 and X_G .

~~$X_F \neq \emptyset$~~ $\emptyset$

Defn $T: X_F \rightarrow X_G$ by letting

$$T(\partial^\alpha F(w)) = \partial^\alpha G(w)$$

and extend T linearly to X_F^0 .

$$\langle \partial^\alpha F(w), \partial^\beta F(w) \rangle = \langle G^\alpha \partial^\beta G(w), \partial^\beta G(w) \rangle.$$

$$\sum a_j \partial^{\alpha_j} F(w)$$

$$\equiv \sum \tilde{a}_j \partial^{\alpha_j} F(w) \quad \text{hence} \quad \|\sum a_j \partial^{\alpha_j} F(w)\|^2 = \|\sum a_j \partial^{\alpha_j} G(w)\|^2.$$

$$\Rightarrow \sum a_j \partial^{\alpha_j} G(w)$$

$$= \sum \tilde{a}_j \partial^{\alpha_j} G(w)$$

hence $\sum a_j \partial^{\alpha_j} F(w) = 0 \Leftrightarrow \sum a_j \partial^{\alpha_j} G(w) = 0$

$\therefore T$ is a well-defn $\checkmark$

⑧

only isomts for $X_F^0 \rightarrow X_G^0$. ~~with the~~

They $\partial^\alpha G(n)$ to $\partial^\alpha F(n)$.

The T can be linearly extended to an isomorphism for X_F to X_G .

$$T(F(z)) = \sum_{(\alpha \in \mathbb{N})} \frac{T(\partial^\alpha F(n))}{\alpha!} z^\alpha = \sum \frac{\partial^\alpha G(n)}{\alpha!} z^\alpha = G(z)$$

Def $[T](v) = [Tx]$ for $x \in X_F$. Then

$$[T] \circ [F(z)] = [T \circ F(z)] = [G(z)].$$

$[T]$ is a one-to-one and isomorphism for (9)
 (X_F) to (X_G) . By the uniqueness of f_u ,

$$\text{we get } [T] \circ [F(z)] = [G(z)]. \text{ Done!}$$

Calabi's extension along curves. ~~real analytic~~

Theorem: Let (M, ω) be an n -dimensional Kähler manifold with a real analytic Kähler form ω .

Let $U \subset M$ be a neighborhood of $z^0 \in M$ and let

$f: U \rightarrow \mathbb{C}^n$ be a local holomorphic ~~embed~~ embedding.

Then for any continuous $\gamma: [0, 1] \rightarrow M$ with

$\exists \gamma(0) = z^0 \in U$, f extends holomorphically as a local isometric embedding.

Proof. $\exists$ an open cover $\{U_j\}_{j=1}^n$ of U such that $U_j \cap U_{j+1} \neq \emptyset$ and a holomorphic partition of unity $\phi_0, \dots, \phi_n \in C^\infty$ s.t.

$$\sum_{j=1}^n \phi_j = 1, \quad \phi_j|_{U_j} = 1, \quad \phi_j|_{U_{j+1}} = 0, \quad \phi_j|_{U_{j+2}} = 0, \dots$$

(10)

pf: we assume $c \in (0, 1]$ to be such that f extends

along $\gamma|_{[0, c]}$ but not across $\gamma(c)$.

Let $t_0 \leq c$ but $t_0 < c$ s.t. $|\dot{\gamma}(t)| \leq \frac{1}{n} \delta_1^2$ for $\delta_1 < c$.

Assume $w = (i\partial\bar{\partial} \varphi)(z, \bar{z})$, $r(t) = 0$.

With a good coordinates, we assume

$$u(z, \bar{z}) = |z|^2 + o(|z|^2) \quad \varphi_{\text{Hilb}}$$

write $\varphi = e^u$ or $w = i\partial\bar{\partial} \log \varphi$.

$\varphi(z, \bar{z})$ is hol in $D(r(t_0), 3\delta) \times D(\overline{r(t_0)}, 3\delta)$.

$$\varphi \neq 0.$$

~~$\varphi(z, \bar{z})$~~ $\Rightarrow$ Taylor expansion of $\log \varphi$ at $r(t_0)$.

$$\log \varphi = \log \left(\varphi(r(t_0)) + \sum_{\alpha} \frac{\varphi'_{\alpha}}{\alpha!} (z - r(t_0))^{\alpha} + \sum_{\alpha, \beta \geq 0} \tilde{b}_{\alpha\beta} (z - r(t_0))^{\alpha} (\bar{z} - \overline{r(t_0)})^{\beta} \right)$$

$$= \log h_0(z) + \log h_0(\bar{z}) + \log \left(1 + \sum_{\alpha, \beta \geq 0} \tilde{b}_{\alpha\beta} (z - r(t_0))^{\alpha} (\bar{z} - \overline{r(t_0)})^{\beta} \right)$$

where $h_0(z) \in \text{hol}(D(r(t_0), 2\delta))$,

$$|\tilde{b}_{\alpha\beta}| \leq C_{\alpha\beta} |\delta|^{-\alpha-\beta}.$$

(11)

now assume (F) is the extension of the original map at $r(t_0)$. write after a linear isometry

$$F = (1, f_2(z), \dots, f_j(z), \dots), \quad f_j(r(t_0)) = 0, \quad j \geq 2$$

Since $w(z) = F^*(w_5 + \dots)$

$$i\partial\bar{\partial} \log \varphi = i\partial\bar{\partial} \log \left(1 + \sum_{j=2}^{\infty} |f_j(z)|^2 \right).$$

Here ~~by~~ $\sum_{|k| \geq 0} b_{\alpha, \bar{\alpha}} (2 - r(k))^\alpha \overline{(2 - r(k))}^\beta$

$$= \sum_{|k| \geq 0} f_{1k}(z) \overline{f_{1k}(z)} \quad \text{near } r(k)$$

Assume $r(k) \geq 0$:

$$\sum_{\alpha, \beta \geq 0} b_{\alpha, \bar{\alpha}} z^\alpha \bar{z}^\beta = \sum f_{1k}(z) \cdot \overline{f_{1k}(z)}$$

$$b_{\alpha, \bar{\alpha}} = \sum_{|k| \geq 0} \left| \frac{1}{\alpha!} \partial^\alpha f_{1k}(0) \right|^2$$

$$|b_{\alpha, \bar{\alpha}}| \leq C_1 \delta_1^{-2|\alpha|}$$

$$\left| \frac{1}{\alpha!} \partial^\alpha f_{1k}(0) \right| \leq |b_{\alpha, \bar{\alpha}}| \leq C_1 \delta_1^{-2|\alpha|} \quad ((12))^p$$

$$\left| \frac{1}{\alpha!} \partial^\alpha f_{1k}(0) \right| \leq \sqrt{C_1} \delta_1^{-|\alpha|}$$

$$f_{1k}(z) = \sum_{|\alpha| \geq 0} \frac{\partial^\alpha f_{1k}(0)}{\alpha!} z^\alpha$$

Converges uniformly on $D(0, \delta_1)$

extends to $\mathbb{P}^n$

converges in

$D(0, \delta_1)$

write the extension as

$\tilde{f}_k$

$$\sum_{k=1}^{\infty} |\hat{f}_k(z)|^2 = \sum_{k=2}^{\infty} \left| \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} \partial^\alpha f_{k(r)}(z) z^\alpha \right|^2$$

$$\leq \sum_{k=2}^{\infty} \sum_{|\alpha| \geq 0} \left| \frac{1}{\alpha!} \partial^\alpha f_{k(r)}(z) z^\alpha \right| \cdot \sum_{|\alpha| \geq 0} |z^\alpha|.$$

$$\leq C_2 \sum_{|\alpha| \geq 0} \sum_{k=2}^{\infty} \left| \frac{1}{\alpha!} \partial^\alpha f_{k(r)}(z) z^\alpha \right| |z^\alpha|.$$

$$\leq C_2 \sum_{|\alpha| \geq 0} |k_\alpha| |z|^\alpha$$

$$\leq C_2 \sum_{|\alpha| \geq 0} S_1^{-2|\alpha|} |z|^\alpha$$

(13)'

The $\sum |\hat{f}_k(z)|^2$ converges in

compact subsets of $D(r_0, S_1^2)$. Hence

f extends to $\bar{D}(r_1, S_1^2) \setminus \{r_1\}$.

Conclude.

~~Here~~
 ~~$1 \leq k \leq n$~~

most important example for isometric
embedding $\rightarrow$ the Bochner-Bergman
map.

Let $\Omega \subset \mathbb{C}^n$ be a domain &
 $\{\varphi_j\}_{j=1}^N$ an o.n.b. of $A^2(\Omega)$

$$(112) \quad \mathcal{B}: \Omega \rightarrow \mathbb{P}^N$$

$$z \mapsto (\varphi_1(z), \varphi_2(z), \dots)$$

$(\varphi_j)'$

$$\text{then } \mathcal{B}^*(\omega_{\mathbb{P}^N}) = \omega_{\text{Berg}} \\ = \frac{1}{2\pi} \sum |\varphi_j|^2.$$

~~Here any~~

if (112) ~~is a map~~, base point free
and separate directions

Cor: one can compute $K_{\mathbb{P}^N}(V) = 2$.

$$\text{Cor: } K_{\omega_{\text{Berg}}}(V) \leq K_{\mathbb{P}^N}(V) \leq 2.$$

could have no lower bound.