Introduction to Hilbert Space Methods in Complex Analysis and Geometry

Notes for a minicourse given at the University of Wisconsin - Madison

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Preface

The following is a set of notes for a minicourse on Hilbert space techniques in complex analytic geometry, presented at the University of Wisocnsin-Madison in late June of 2024. The minicourse consisted of four lectures, which were essentially summaries of much of what is contained in this text.

As usual, such a course has to have a target audience. We assume that the student attending the minicourse, or for that matter the reader of these notes, has a thorough familiarity with much of undergraduate mathematics, including calculus and algebra (mostly linear, some multilinear, and very little basic abstract algebra), as well as the following topics, which might not be covered in all undergraduate programs in the US.

a. Basic point set topology.

b. Smooth manifolds.

c. The definition and some elementary properties of smooth vector bundles, including the tangent and cotangent bundle. (We do not assume a knowledge of characteristic classes, or of the theory of connections.)

d. Elementary properties (essentially just the definitions) of smooth vector fields, differential forms, and smooth sections of vector bundles.

e. Certainly a few other things I have assumed without realizing.

In these notes we invoke the complex summation convention of summing over the same indices when one is upper and the other is lower. Rather than being just convenient, this convention becomes useful in identifying various dualities that exist between geometrically complicated quantities.

Chapter 1

The Backdrop

1.1 Complex Manifolds and Maps

In this subsection we establish the basic definitions of complex analysis in the manifold setting. The reader who has never seen any of the contents of this chapter is likely not yet at the stage of preparation assumed in this text. Accordingly, it is probably wise for the average reader to skim, or perhaps skip, this section.

1.1.1 Complex Manifolds

Definition

A complex manifold is a real manifold whose transition functions are holomorphic. More precisely,

1.1.1 Definition. A topological space X is a complex manifold if there is an open cover $\{U_j\}_{j\in J}$ of X and homeomorphisms $\phi_j : U_j \to V_j \subset \mathbb{C}^{n_j}$, called *holomorphic coordinate charts*, such that for all $i, j \in J$ the the maps

$$\phi_{ij} := \phi_i \circ \phi_j : V_j \cap \phi_j \circ \phi_i(V_i) \to V_i,$$

called *transition functions*, are holomorphic. ◇

Remark. Sometimes the inverse maps $\phi_j^{-1} : V_j \to U_j$ are called coordinate charts. Usually this ambiguity does not cause any confusion. ◇

A collection of (holomorphic) coordinate charts $\mathscr{A} := \{U_j, \phi_j : U_j \to \phi_j\}_{j\in J}$ is called a (*holomorphic*) *atlas* if $\bigcup_j U_j = X$. Two atlases are said to be compatible if their union is also an atlas. An equivalence class of atlases is a directed set, i.e., any two atlases are contained in a third atlas, namely their union. Thus every equivalence class of atlases contains a *maximal atlas*, i.e., an atlas that is not properly contained in any other atlas.

Maximal atlases are huge collections of coordinate charts, and one cannot reasonably characterize such a collection. However, specifying any atlas automatically uniquely determines a maximal atlas.

We leave the proof of the next statement to the reader.

1.1.2 PROPOSITION/DEFINITION. *The function* $\dim_{\ell oc} : X \to \mathbb{Z}$ *assigning to each* $x \in X$ *the nonnegative integer* n_j *if and only if* $x_j \in U_j$ *is locally constant. If* X_μ *is a connected component of* X *then the integer* $n_\mu := \dim_{\ell oc}(X_\mu)$ *is called the* dimension *of* X_μ.

In most situations considered in this text the manifold in question is connected.

ORIENTABILITY

Suppose $\Omega \subset \mathbb{C}^n$ is an open set and $F : \Omega \to \mathbb{C}^n$ is a holomorphic map. Let us denote by $F_{\mathbb{R}}$ the map obtained from F after identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via

$$(z^1, ..., z^n) \mapsto (\text{Re } z^1, \text{Im } z^1, ..., \text{Re } z^n, \text{Im } z^n) = (x^1, y^1, ..., x^n, y^n).$$

Noticing that the complex differential 1-forms $dz^i = dx^i + \sqrt{\text{-}1}dy^i$ satisfy

$$\left(\frac{\sqrt{\text{-}1}}{2}\right)^n dz^1 \wedge d\bar{z}^1 \wedge ... \wedge dz^n \wedge d\bar{z}^n = dx^1 \wedge dy^n \wedge ... \wedge dx^n \wedge dy^n,$$

we see that the functions $\det dF$ and $\det dF_{\mathbb{R}}$ defined respectively by

$$(\det dF)dz^1 \wedge ... \wedge dz^n := F^*(dz^1 \wedge ... \wedge dz^n)$$

and

$$(\det F_{\mathbb{R}})dx^1 \wedge dy^n \wedge ... \wedge dx^n \wedge dy^n := F^*(dx^1 \wedge dy^n \wedge ... \wedge dx^n \wedge dy^n)$$

satisfy

$$\begin{aligned}
(\det dF_{\mathbb{R}})\, dx^1 \wedge dy^n \wedge ... \wedge dx^n \wedge dy^n &:= F_{\mathbb{R}}^* \left(dx^1 \wedge dy^n \wedge ... \wedge dx^n\right) \\
&= F^* \left(\left(\frac{\sqrt{\text{-}1}}{2}\right)^n dz^1 \wedge d\bar{z}^1 \wedge ... \wedge dz^n \wedge d\bar{z}^n \right) \\
&= \frac{\sqrt{\text{-}1}^{n^2}}{2^n} F^* \left(dz^1 \wedge ... \wedge dz^n \wedge d\bar{z}^1 \wedge ... \wedge d\bar{z}^n\right) \\
&= |\det dF|^2 \left(\left(\frac{\sqrt{\text{-}1}}{2}\right)^n dz^1 \wedge d\bar{z}^1 \wedge ... \wedge dz^n \wedge d\bar{z}^n \right).
\end{aligned}$$

It follows that the determinants of the real derivatives of the transition functions of a holomorphic atlas are all positive, i.e., that every holomorphic atlas is oriented. In particular, we have proved the following proposition.

1.1.3 PROPOSITION. *Every complex manifold is orientable.*

REMARK. In complex dimension 1 Proposition 1.1.3 has a sort of converse, namely that every oriented surface admits a (not necessarily unique) holomorphic atlas. This converse implication fails to hold in higher dimensions; there are many examples of oriented manifolds that are not complex manifolds. ◇

Some examples

1.1.4 Example (Vector spaces). The simplest example of a complex manifold is a complex vector space. If V is a complex vector space, a choice of basis $\mathbf{v} = \{v_1, ..., v_n\}$ defines a bijection

$$\psi_{\mathbf{v}} : \mathbb{C}^n \ni z = (z^1, ..., z^n) \mapsto z^1 v_1 + \cdots + z^n v_n \in V,$$

and we declare the inverse $\phi_{\mathbf{v}} := \psi_{\mathbf{v}}^{-1}$ to be a holomorphic atlas. Evidently the dimension of V as a complex manifold is its dimension as a complex vector space. ◇

1.1.5 Example (Affine Spaces). The next simplest example of a complex manifold is a complex affine space A, i.e., a set equipped with a faithful action of a finite-dimensional complex vector space V (viewed as an Abelian group), with only one orbit. The action is denoted by

$$A \times V \ni (a, v) \mapsto a + v \in A.$$

In terms of this notation, for any two points $a_1, a_2 \in A$ there is a unique vector $v \in V$ such that $a_2 = a_1 + v$. Thus there is a bijection between A and V, but there is no canonical bijection. The space A is topologized be declaring that a subset $U \subset A$ is open if for every $a \in U$ there is an open neighborhood B of the origin in V such that $\{a + v \ ; \ v \in B\} \subset U$.

The vector space V is called the space of *affine translations* of A, and also the *underlying vector space of* A.

Once a point $o \in A$ is chosen, there is a canonical bijection

$$V \ni v \mapsto o + v.$$

Since vector space topology is invariant under translations, this canonical bijection is a homeomorphism.

A holomorphic atlas is specified by choosing a point $o \in A$ and a basis $\mathbf{v} := \{v_1, ..., v_n\}$ for V. Since any point $a \in A$ is of the form $a = o + z^1 v_1 + \cdots + z^n v_n$ for some $z = (z^1, ..., z^n) \in \mathbb{C}^n$, the map

$$\phi_{o,\mathbf{v}}(a) := z$$

defines a 'global' coordinate chart, and a maximal atlas $\mathscr{A}_{o,\mathbf{v}}$. ◇

Remark. In Example 1.1.5 we have assumed that the vector space V is finite-dimensional. There is also a notion of infinite-dimensional affine space, in which case V must be a topological vector space. The topologies of infinite-dimensional vector spaces are much more complicated than their finite-dimensional cousins. ◇

1.1.6 Example (Open subsets and Cartesian products). Any open subset of a complex manifold is a complex manifold, when equipped with the relative topology. Cartesian products of complex manifolds are complex manifolds. ◇

1.1.7 Example (Complex Projective Spaces). Let V be a finite dimensional complex vector space of complex dimension n. Denote by $\mathbb{P}(V)$ the set of 1-dimensional subspaces of V, i.e., lines through the origin in V. Since every $v \in V - \{0\}$ determines a unique line through the origin, there is a surjective map $\pi : V - \{0\} \to \mathbb{P}(V)$. We topologize $\mathbb{P}(V)$ by declaring that a set $U \subset \mathbb{P}(V)$ is open if and only if $\pi^{-1}(U)$ is an open subset of $V - \{0\}$. In other words, we endow $\mathbb{P}(V)$ with the coarsest topology with respect to which π is continuous.

If we take a basis $\mathbf{v} = \{v_1, ..., v_n\}$ for V, then the set

$$\mathbb{S}_{\mathbf{v}} := \left\{ z^1 v_1 + \cdots + z^n v_n \ ; \ |z^1|^2 + \cdots + |z^n|^2 = 1 \right\}$$

satisfies $\pi(\mathbb{S}_{\mathbf{v}}) = \mathbb{P}(V)$, and hence $\mathbb{P}(V)$ is a compact topological space.

For every non-zero linear functional $\ell \in V^* - \{0\}$ we can define a subset

$$U_\ell := \{P \in \mathbb{P}(V) \ ; \ \ell(v) \neq 0 \text{ for some (and hence any) } v \in P - \{0\}\}.$$

Writing $V_\ell := \{x \in V \ ; \ \ell(x) = 1\}$, consider the map $\phi_\ell : U_\ell \to V_\ell$ defined by

$$\phi_\ell(P) := \pi^{-1}(P) \cap V_\ell.$$

Since $\ell(v) \neq 0$ for any $v \in P$, ϕ_ℓ is well-defined. Moreover, if $\phi_\ell(P) = \phi_\ell(Q)$ then $P = Q$, so that ϕ_ℓ is injective. Finally, if $x \in V_\ell$ then for every $v \in \pi(x)$ we have $v = \lambda x$ for some $\lambda \in \mathbb{C} - \{0\}$, and hence $\ell(v) \neq 0$. Thus $\pi(v) \in U_\ell$ and $\phi_\ell(\pi(v)) = v$, so that ϕ_ℓ is also surjective, hence bijective.

It is easy to see that for every $\ell \in V^*$ the set $V_\ell \subset V$ is an affine space, and that

$$\bigcup_{\ell \in V^* - \{0\}} U_\ell = \mathbb{P}(V).$$

We leave it as an exercise to show that the maps

$$\phi_{\ell'}^{-1} \circ \phi_\ell : V_\ell \to V_{\ell'}$$

are biholomorphic, and hence $\mathbb{P}(V)$ is a complex manifold. ◇

Remark. When the vector space is the complex Cartesian space $V = \mathbb{C}^{n+1}$ we shall use the notation

$$\mathbb{P}_n := \mathbb{P}(\mathbb{C}^{n+1})$$

for its projectivization.

1.1.8 Example (Hopf Manifolds). Let V be a complex vector space of dimension n and let $c \in \mathbb{C}$ be a complex number of modulus $|c| > 1$. Two vectors $v, w \in V - \{0\}$ are said to be equivalent if $v = c^n w$ for some $n \in \mathbb{Z}$. The quotient space

$$\mathrm{Hopf}_c(V)$$

is topologized with the coarsest topology for which the quotient map $\pi : V - \{0\} \to \mathrm{Hopf}(c)$ is continuous.

Fix a basis $\mathbf{v} = \{v_1, ..., v_n\}$ for V. Since

$$\pi \left(\bigcup_{1 \leq r \leq |c|} r\mathbb{S}_{\mathbf{v}} \right),$$

$\mathrm{Hopf}_c(V)$ is a compact topological space.

We can equip $\mathrm{Hopf}_c(V)$ with a holomorphic atlas as follows. The open cover consists of the two open sets

$$U_1 := \pi \left(\bigcup_{1 < r < |c|} r\mathbb{S}_{\mathbf{v}} \right) \quad \text{and} \quad U_2 := \pi \left(\bigcup_{\frac{1+|c|}{2} < r < \frac{1+3|c|}{2}} r\mathbb{S}_{\mathbf{v}} \right),$$

where $\mathbf{v} = \{v_1, ..., v_n\}$ is some chosen basis for V. The map $\pi : V - \{0\} \to \mathrm{Hopf}_c(V)$ restricts to bijections on the open sets

$$V_1 := \bigcup_{1 < r < |c|} r\mathbb{S}_{\mathbf{v}} \quad \text{and} \quad V_2 := \bigcup_{\frac{1+|c|}{2} < r \leq \frac{1+3|c|}{2}} r\mathbb{S}_{\mathbf{v}}$$

respectively, so we set $\phi_i := \left(\pi|_{V_i}\right)^{-1}$, $i = 1, 2$. The transition functions are easily seen to be holomorphic, since $\phi_1^{-1} \circ \phi_2 = \mathrm{Id}$.

It is easy to see that $\mathrm{Hopf}_c(V)$ is homeomorphic to $S^1 \times S^{2n-1}$. It follows that $\mathrm{Hopf}_c(V)$ cannot be a symplectic manifold. Indeed, for topological reasons every closed 2-form on $\mathrm{Hopf}_c(V)$ is exact, but on a compact manifold a symplectic form, i.e., a closed non-degenerate 2-form, is never exact. ◇

1.1.9 Example (Complex conjugate manifold). Let X be a complex manifold with maximal atlas $\mathscr{A} := \{\phi_\alpha : U_\alpha \to V_\alpha\}$. We define a new atlas

$$\mathscr{A}^\dagger := \{C \circ \phi_\alpha : U_\alpha \to C(V_\alpha)\},$$

where $C : \mathbb{C}^n \ni z \mapsto \bar{z} \in \mathbb{C}^n$ is the map of complex conjugation. It is left to the reader to check that $\mathscr{A}^\dagger$ is a holomorphic maximal atlas, and moreover that this atlas is not compatible with the atlas $\mathscr{A}$. The smooth manifold X with the atlas $\mathscr{A}^\dagger$ shall be denoted $X^\dagger$. ◇

To have more examples of complex manifolds, we shall need to define functions and maps.

1.1.2 Functions and Maps

Definition

1.1.10 Definition. Let X and Y be complex manifolds.

i. A function $f : X \to \mathbb{C}$ is said to be *holomorphic* if for every holomorphic coordinate chart $\phi : U \to V \subset \mathbb{C}^n$, $U \overset{\text{open}}{\subset} X$, the function $f \circ \phi^{-1} : V \to \mathbb{C}$ is holomorphic. We write $f \in \mathcal{O}(X)$.

ii. A map $F : Y \to X$ is said to be *holomorphic* if for every $\phi : U \to V \subset \mathbb{C}^n$, $U \overset{\text{open}}{\subset} X$, the function $\phi \circ F|_{F^{-1}(U)}$ is holomorphic. We write $F \in \mathcal{O}(X, Y)$.

iii. A map $F : X \to Y$ is said to be biholomorphic if it is holomorphic, bijective, and the inverse map $F^{-1} : Y \to X$ is also holomorphic.

1.1.11 Proposition. *A compact connected complex manifold X has no non-constant holomorphic functions.*

Proof. Let $f : X \to \mathbb{C}$ be a holomorphic function. By continuity the set

$$\mathscr{M}_f := \left\{ x \in X \ ; \ |f(x)| = \sup_X |f| \right\}$$

is closed, and by compactness it is nonempty. To complete the proof, it suffices to show that $\mathscr{M}_f$ is open. Toward that end, let $x \in \mathscr{M}_f$. Choose a holomorphic coordinate chart $\phi : U \to V \subset \mathbb{C}^n$ such that $p \in V$. Then the function $f \circ \phi^{-1} : V \to \mathbb{C}$ is holomorphic and $|f \circ \phi^{-1}|$ has in interior maximum in V. By the Maximum Principle f is constant on the connected component U_x of U containing x. In particular, $U_x \subset \mathscr{M}_f$, so $\mathscr{M}_f$ is open. □

1.1.12 Proposition. *Let X and Y be complex manifolds. If $F : X \to Y$ is a bijective holomorphic map then F^{-1} is holomorphic.*

Proof. The claim to be proved is local, so we may assume that X and Y are open connected subsets of $\mathbb{C}^n$ containing the origin. We argue by induction on the dimension n.

In dimension 1 the argument is as follows. If $F(z) = z^m g(z)$ with $g(0) \neq 0$ then perhaps after shrinking X we may write $g(z) = e^{mG(z)}$ for some holomorphic function $G \in \mathcal{O}(X)$. Making the change of coordinates $w = ze^{G(z)}$, which is invertible by the holomorphic version Inverse Function Theorem, we may assume that $F(w) = w^m$. This map is bijective if and only if $m = 1$, and evidently the inverse is holomorphic.

Next suppose the proposition holds in all dimensions $k \leq n - 1$. If the linear map $dF(o)$ has rank n then one concludes by the holomorphic Inverse Function Theorem, so assume the rank of $dF(o)$ is $k \leq n - 1$. Perhaps after a permutation of the coordinates in the ambient '$\mathbb{C}^n$'s containing the domains U and V, we may assume that the matrix

$$\left(\frac{\partial F_i}{\partial z^j}(o) \right)_{i,j=1}^{k}$$

is invertible. Hence the derivative $\partial f(o)$ of map

$$f(z^1, ..., z^n) := (F^1(z), ..., F^k(z), z^{k+1}, ..., z^n)$$

is invertible. By the holomorphic Inverse Function Theorem f maps a small neighborhood U of o in $\mathbb{C}^n$ biholomorphically onto its image $V := F(U)$.

Now consider the holomorphic map

$$g := F \circ f^{-1} : V \to Y.$$

The map g carries $V \cap \{o\} \times \mathbb{C}^{n-k}$ to $Y \cap \{o\} \times \mathbb{C}^{n-k}$ injectively, and by construction the derivative of $g|_{V\cap\{o\}\times\mathbb{C}^{n-k}}$ is not invertible. By the induction hypothesis $k = 0$. In particular, we see that if $dF(o)$ is not invertible then it is identically zero. Equivalently, if $\det dF(o) = 0$ then $dF(o) = 0$.

It follows that F maps every connected component of $\{\det dF = 0\}$ to a single point. Since F is a bijection, we conclude that the zero set of the holomorphic function $\det dF$ is a closed discrete subset. But if $n \geq 2$ then the zero set of a holomorphic function cannot be isolated unless it is empty. This contradiction completes the proof. □

1.1.13 Example.

i. Every coordinate chart in a holomorphic atlas is holomorphic by definition.

ii. With respect to the holomorphic atlases defined in Examples 1.1.4 and 1.1.5, an affine space is biholomorphic to its underlying vector space.

iii. The projection maps $\pi : V - \{0\} \to \mathbb{P}(V)$ and $\pi : V - \{0\} \to \mathrm{Hopf}_c(V)$ of Examples 1.1.7 and 1.1.8 respectively are holomorphic. The latter map is locally biholomorphic, but the former is not. ◇

Recall that if M_1 and M_2 are manifolds, $\pi : E \to M_2$ is a vector bundle and $f : M_1 \to M_2$ is a map then the pullback $f^*E \to M_1$ is the vector bundle defined by the incidence relation

$$f^*E := \{(x, v) \in M_1 \times E \ ; \ \pi(v) = f(x)\} \subset M_1 \times E,$$

and the vector bundle projection $f^*E \to M_1$ is obtained by restriction to f^*E of the Cartesian projection $M_1 \times E \to M_1$.

1.1.14 Definition (Differentials and derivatives). Let X and Y be complex manifolds.

a. If $F \in \mathcal{O}(X, Y)$ the *differential* of F is the vector bundle map $DF : T_X \to F^*T_Y$ defined by

$$DF(x, v) := \left(x, \ \frac{d}{dt}\Big|_{t=0} F(\gamma(t))\right)$$

for any curve $\gamma : I \to X$ such that $0 \in I$ and $\dot{\gamma}(0) = v$.

b. The *derivative* of F is the section dF of $T_X^* \otimes F^*T_Y \to X$ corresponding to the vector bundle map DF via the identification $\mathrm{Hom}(E, F) \cong E^* \otimes F$. ◇

If $Y = \mathbb{C}$ then T_Y is the trivial vector bundle, and hence so is F^*T_Y. Thus for a holomorphic function $f \in \mathcal{O}(X)$ the derivative *df* is a section of the cotangent bundle T_X^*, i.e., a 1-form.

Submanifolds

There are two standard notions of complex submanifolds. Recall that a map $F : M \to N$ is said to be an immersion (resp. submersion) if the differential DF is a fiberwise-injective (resp. surjective) vector bundle map.

1.1.15 Definition. Let X and S be complex manifolds.

1. If $F : S \to X$ is an injective holomorphic immersion, we say that $F(S)$ is an *immersed complex submanifold*.
2. If $F : S \to X$ is an injective holomorphic immersion that is homeomorphic onto its image, we say that $F(S)$ is an *embedded complex submanifold*. $\diamond$

Remark. If the complex manifold S is already realized as a subset of X, the injective immersion F in question is taken to be the natural inclusion of subsets. $\diamond$

An immersed complex submanifold $F(S) \subset X$ is embedded if and only if the injective immersion F is a proper map, i.e., the inverse image of compact sets is compact. (This statement is true in the smooth category as well.) In these notes, we shall be dealing only with embedded submanifolds. We shall therefore drop the adjective *embedded*.

1.1.16 Proposition. *Let $S \subset X$ be a subset. Then the following are equivalent.*

a. With respect to the subspace topology, S is a complex submanifold of X.

b. Every point $x \in X$ is contained in a neighborhood $U \subset X$ and there are holomorphic functions $f_1, ..., f_k$ such that

$$U \cap S = \{y \in X \ ; \ f_1(y) = ... = f_k(y) = 0\} \quad \textit{and} \quad df_1 \wedge ... \wedge df_k(x) \neq 0.$$

The proof of Proposition 1.1.16 is an application of the holomorphic implicit function theorem. We omit the proof here, but the reader is welcome to supply it.

Two immediate consequences of Proposition 1.1.16 are the following.

- Submanifolds are closed subsets.
- Locally, every holomorphic function on a submanifold is the restriction of a holomorphic function defined in the ambient space.

1.1.17 Example (Tautological line bundle / blow-up of the origin). Let V be a complex vector space. We already know that $V \times \mathbb{P}(V)$ is a complex manifold. We can define a submanifold $\mathbb{U}(V) \subset V \times \mathbb{P}(V)$ by the incidence relation

$$\mathbb{U}(V) := \{(v, P) \in V \times \mathbb{P}(V) \ ; \ v \in P\}.$$

To prove that $\mathbb{U}(V)$ is a submanifold, one should work with the coordinate charts for $\mathbb{P}(V)$ defined in Example 1.1.7. We leave this work to the reader, as well as the verification that $\dim_{\mathbb{C}} \mathbb{U}(V) = \dim_{\mathbb{C}}(V)$.

Now consider the two projection maps $\pi_1 : \mathbb{U}(V) \to V$ and $\pi_2 : \mathbb{U}(V) \to \mathbb{P}(V)$ obtained by restriction to $\mathbb{U}(V)$ of the Cartesian projections $V \times \mathbb{P}(V) \to V$ and $V \times \mathbb{P}(V) \to \mathbb{P}(V)$ respectively. These maps are holomorphic, and both play important roles in complex geometry.

Let us begin with the second map $\pi_2 : \mathbb{U}(V) \to \mathbb{P}(V)$. The inverse image of a point $P \in \mathbb{P}(V)$ consists of all point of V lying in this 1-dimensional subspace. Thus the fibers of π_2 are complex lines. The more common notation for the total space of this fibration is $\mathcal{O}_V(-1)$, and

$$\pi_2 : \mathcal{O}_V(-1) \to \mathbb{P}(V)$$

is the first nontrivial example of a holomorphic line bundle, a concept we discuss in the next section.

Turning to the map $\pi_1 : \mathbb{U}(V) \to \mathbb{P}(V)$, we note that if $v \in V$ then $\pi_1^{-1}(\{v\})$ consists of all lines through the origin in V passing through v. If $v \neq 0$ then there is exactly one such line, while if $v = 0$ then every line through 0 contains v. Thus $\pi_1|_{\mathbb{U}(\mathbb{P}(V) - \pi_1^{-1}(\{0\})} : \mathbb{U}(V) - \pi_1^{-1}(\{0\}) \to V - \{0\}$ is a holomorphic bijection (and thus is biholomorphic), while $\pi_1^{-1}(\{0\})$ is a copy of $\mathbb{P}(V)$. In this setting the more common notation for the domain of π_1 is $\mathrm{B}\ell_o(V)$— this complex manifold is called the *blowup of the origin in V*— and the map

$$\pi_1 : \mathrm{B}\ell_o(V) \to V$$

is called the blow-down map.

It is worth observing that the construction can be carried out V replaced by an open neighborhood of the origin in V. In this way, one can import the blow-up construction to any complex manifold by working in a holomorphic chart, so the blow-up of a point can be defined on any complex manifold. ◇

Exercises

1. Show that an open subset of a complex manifold is a complex manifold.

2. This exercise concerns Example 1.1.5.

   a. Show that for any two points $a, a' \in A$ and any invertible linear transformation $A \in \mathrm{GL}(V)$ the map $\phi_{a',\mathbf{v}}^{-1} \circ \phi_{a,A\mathbf{v}} : A \to A$ is biholomorphic.

   b. Show that any two affine spaces are biholomorphic.

   c. Suppose $o, o' \in A$ and $\mathbf{v}, \mathbf{v}'$ are two bases for the space of translations V of A. Show that the maximal atlases $\mathscr{A}_{o,\mathbf{v}}$ and $\mathscr{A}_{o',\mathbf{v}'}$ are the same.

3. Let V be a complex vector space and let $\ell \in V^*$ be a non-zero linear functional.

   a. What is the dimension of $\mathbb{P}(V)$?

   b. Compute the transition functions defined in Example 1.1.7.

   c. Show that $\{P \in \mathbb{P}(V) \ ; \ \ell(v) = 0 \text{ for all } v \in P\}$ is also a projective space. What is its dimension? What is its complement in $\mathbb{P}(V)$?

   d. Suppose V and V' are two complex vector spaces. Under what conditions are $\mathbb{P}(V)$ and $\mathbb{P}(V')$ biholomorphic?

4. A 2-form α on a smooth manifold M is said to be non-degenerate at $p \in M$ if the only vector $v \in T_{M,p}$ satisfying

$$\alpha(v, w) = 0 \quad \text{for all } w \in T_{M,p}$$

   is the zero-vector $v = 0$. A 2-form on M is said to be non-degenerate if it is non-degenerate at every point of M. A symplectic form is a closed non-degenerate 2-form.

   Let α be a non-degenerate 2-form on a compact real manifold M.

   a. Show that M is even-dimensional.

   b. Show that if α is symplectic then α is not exact.

5. Let M and Σ be smooth manifolds and suppose there exists an injective immersion $f : \Sigma \to M$. Prove that Σ is homeomorphic to $f(\Sigma)$ with its subspace topology if and only if the map f is proper.

1.2 Holomorphic Vector Bundles

Definitions

A holomorphic vector bundle of rank r is a triple $(E, X, \pi : E \to X)$ such that

(i) E and X are complex manifolds and the map π is holomorphic,

(ii) the fibers $E_x := \pi^{-1}(x)$ of π have vector space structures such that the map

$$E \times \mathbb{C} \ni (v, \lambda) \mapsto \lambda \cdot v \in E$$

is holomorphic, and

(iii) each $p \in X$ is contained in an open set U on which there are holomorphic maps $e_1, ..., e_r : U \to E$ such that

$$\pi e_i(x) = x \quad \text{and} \quad \text{span}_{\mathbb{C}}\{e_1(x), ..., e_r(x)\} = E_x \text{ for all } x \in U.$$

Such a collection of maps $\{e_1, ..., e_r\}$ is called a *frame for E over U*.

Observe that if $\{e_i\}$ and $\{\tilde{e}_i\}$ are two frames defined over the same open set U, then there are holomorphic functions $g_i^j \in \mathcal{O}(U)$ such that $\tilde{e}_i = g_i^j e_j$ and $\left(g_i^j(p)\right)_{i,j=1}^r \in GL(r, \mathbb{C})$ for all $p \in U$.

1.2.1 Remark. If "holomorphic" and "complex manifold" are replaced by "smooth (or continuous)" and "smooth manifold (or topological space)" one obtains the definition of complex vector bundle. ◇

The positive integer r above is called the *rank* of $(E, X, \pi : E \to X)$. Evidently the rank is a locally constant function on the base manifold X. We will work only with vectors bundles of constant rank.

A holomorphic vector bundle of rank 1 is called a *holomorphic line bundle*. Holomorphic line bundles play an especially important role in complex analytic and algebraic geometry.

A map of holomorphic vector bundles is a holomorphic map $F : E \to E'$ such that

(i) the diagram

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\text{Id}} & X \end{array}$$

commutes, and

(ii) for each $x \in X$ the map

$$F_x := F|_{E_x} : E_x \to E'_x$$

is linear.

Two vector bundles are isomorphic if there are holomorphic vector bundle maps $F : E \to E'$ and $G : E' \to E$ such that $FG = \mathrm{Id}_{E'}$ and $GF = \mathrm{Id}_E$.

A section s of a holomorphic vector bundle $\pi : E \to X$, i.e., a right inverse for π, is said to be holomorphic (resp. smooth, measurable, etc.) if it is holomorphic (resp. smooth, measurable, etc.) as a map $X \to E$. (In this language, a frame over U is a collection of sections whose values at each point $x \in U$ form a basis for E_x.)

1.2.2 Example (Trivial bundles). The simplest example of a holomorphic vector bundle is the trivial bundle $\pi : X \times \mathbb{C}^r \to X$, where π denotes the projection to the first factor. If a vector bundle $E \to X$ is isomorphic to the trivial bundle, then for any basis $\mathbf{e}_1, ..., \mathbf{e}_r$ of $\mathbb{C}^r$ the isomorphism $F : X \times \mathbb{C}^r \to E$ defines a frame

$$e_i(x) := F(x, \mathbf{e}_i), \quad 1 \le i \le r$$

over the whole of X. Conversely, a global frame for a vector bundle $E \to X$ defines an isomorphism F^{-1}, where F is given by the same formula and then extended fiberwise-linearly. That is to say, if we fix a basis $\mathbf{e}_1, ..., \mathbf{e}_r$ of $\mathbb{C}^n$, we define the isomorphism $F : X \times \mathbb{C}^r \to E$ by

$$F^{-1}(f^i(x)e_i(x)) := (x, f^i(x)\mathbf{e}_i).$$

Thus a vector bundle is isomorphic to the trivial bundle if and only if the vector bundle has a global frame. In particular, every (holomorphic) vector bundle is locally trivial. ◇

1.2.3 Example (New bundles from old, pullbacks, etc).

(i) Observe that if $E \to X$ and $E_o \to X$ are holomorphic vector bundles the so are $E^* \to X$, $E \otimes E_o \to X$ and $E \oplus E_o \to X$. Thus $\mathrm{Sym}^k(E) \to X$ and $\Lambda^k E \to X$ are holomorphic vector bundles, as are all vector bundles obtained from holomorphic vector bundles from fiberwise $\mathbb{C}$-multilinear operations. On the other hand, the complex conjugate bundle $\overline{E} \to X$, which one often meetst, is in general *not* a holomorphic vector bundle.

(ii) In topology (or even category theory) one has the so-called *fiber product*, which generalizes the incidence relation constructions we have now seen several times: Given morphisms $f : X \to Z$ and $g : Y \to Z$ one defines

$$X \times_Z Y := \{(x, y) \in X \times Y \ ; \ f(x) = g(y)\}.$$

There are projection maps $X \times_Z Y \to X$ and $X \times_Z Y \to Y$ given by the restriction to $X \times_Z Y$ of the Cartesian projections $X \times Y \to X$ and $X \times Y \to Y$.

If $\pi : E \to Y$ is a holomorphic vector bundle and $f : X \to Y$ is a holomorphic map then

$$f^*E = E \times_Y X \to X$$

is a holomorphic vector bundle, called the pullback of E by f.

(iii) Given holomorphic vector bundles $E \to X$ and $E' \to Y$, one defines the holomorphic vector bundle
$$E \boxtimes E' = p_X^* E \otimes p_Y^* E' \to X \times Y,$$
where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the Cartesian projections.

1.2.4 Example. A (holomorphic) vector bundle map $F : E \to E'$ can be identified with a (holomorphic) section of the bundle $\mathrm{Hom}(E, E') := E' \otimes E^*$.

In the special case in which $E = E'$, the vector bundle $E \otimes E^*$ always has a globally defined (holomorphic) section, namely the one corresponding to the identity map $E \to E$. In particular, if E has rank 1 then $\mathrm{Hom}(E, E)$ is the trivial line bundle. ◇

Transition functions

Let $E \to X$ be a holomorphic vector bundle. As already pointed out, the choice of a frame over an open set $U \subset X$ yields a vector bundle isomorphism ψ to the trivial bundle $U \times \mathbb{C}^r \to U$, defined by
$$\psi(t^i e_i(x)) = (x, t).$$
With two such frames, and the corresponding isomorphisms ψ and ψ', one can form a map
$$\psi' \circ \psi^{-1} : U \cap U' \times \mathbb{C}^r \to U \cap U' \times \mathbb{C}^r.$$
It is clear from the definitions that
$$\psi' \circ \psi^{-1}(x, t) = (x, g_{U'U}(x)t)$$
for some holomorphic function $g_{U'U} : U \cap U' \to GL(r, \mathbb{C})$. The functions $g_{U'U}$ are called *transition functions*, and they contain all of the information of (the isomorphism class of) the vector bundle.

To be more precise, suppose we cover X by a collection of open sets U_α, $\alpha \in A$, over each of which one has a frame for E. Then we get a collection of holomorphic transition functions
$$\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C}) \ ; \ \alpha, \beta \in A\}.$$
It is easy to see that the conditions
$$g_{\alpha\alpha} = \mathrm{Id} \quad \text{and} \quad g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathrm{Id}, \qquad \alpha, \beta, \gamma \in A$$
on their domains of definition.

Conversely, if one is given an open cover $\{U_\alpha \ ; \ \alpha \in A\}$ of X and a collection of holomorphic functions
$$\mathscr{T} := \{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C}) \ ; \ \alpha, \beta \in A\}$$
that satisfy the so-called *cocycle condition*
$$g_{\alpha\alpha} = \mathrm{Id} \quad \text{and} \quad g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathrm{Id}, \qquad \alpha, \beta, \gamma \in A$$

on their domains of definition. (By taking $\gamma = \alpha$ one sees that $g_{\alpha\beta}^{-1} = g_{\beta\alpha}$.) Then one can define the total space

$$E(\mathscr{T}) := \left(\coprod_{\alpha \in A} U_\alpha \times \mathbb{C}^r \right) / \sim,$$

where

$$U_\alpha \times \mathbb{C}^r \ni (x,t) \sim (y,s) \in U_\beta \times \mathbb{C}^r \quad \Longleftrightarrow \quad x = y \quad \text{and} \quad s = g_{\alpha\beta}(x)t,$$

(the cocycle condition implies that $\sim$ is an equivalence relation) and the map $\pi(\mathscr{T}) : E(\mathscr{T}) \to X$ by $\pi(\mathscr{T})([x,t]) = x$. It is easy to see that $E(\mathscr{T})$ is a holomorphic vector bundle whose transition functions are $\mathscr{T}$. Moreover, if $\mathscr{T}$ is a collection of holomorphic transition functions for $E \to X$ the $E(\mathscr{T}) \to X$ is isomorphic to $E \to X$. Indeed, if we denote by $e_{1\alpha}, ...e_{r\alpha}$ the frame over U_α then the map

$$F([x,t]) := t^1 e_{1\alpha}(x) + \cdots + t^r e_{r\alpha}, \quad x \in U_\alpha(x)$$

yields a well-defined holomorphic vector bundle isomorphism.

On the other hand, if $E \to X$ and $E' \to X$ are isomorphic, they need not have the same transition functions. The reader can confirm that if

$$\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r,\mathbb{C}) \ ; \ \alpha, \beta \in A\} \quad \text{and} \quad \{\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r,\mathbb{C}) \ ; \ \alpha, \beta \in A\}$$

are transition functions for $E \to X$ and $E' \to X$ over the same open cover (if the open covers are locally finite then by taking finite intersections this can always be arranged) then $E \to X$ and $E' \to X$ are isomorphic if and only if there exist maps $\{h_\alpha : U_\alpha \to GL(r,\mathbb{C}) \ ; \ \alpha \in A\}$ such that

$$\tilde{g}_{\alpha\beta} = h_\alpha^{-1} g_{\alpha\beta} h_\beta.$$

It is worth mentioning that if $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ are transition functions for vector bundles E and E' respectively then

(i) $(g_{\alpha\beta}^{-1})^{\text{trans}}$ are transition functions for E^*, and

(ii) $g_{\alpha\beta} \times h_{\alpha\beta}$ are transition functions for $E \times E'$, where $\times = \otimes, \wedge, \odot$ (symmetric product), etc., where $g_{\alpha\beta} \times h_{\alpha\beta}(v \times w) = (g_{\alpha\beta} v) \times (h_{\alpha\beta} w)$.

In general it is not difficult to figure out the transition functions for a vector bundle obtained from other vector bundles through multilinear operations.

The tautological bundle again

1.2.5 Example (The Tautological bundle and its dual, the hyperplane bundle). We have already met the tautological line bundle

$$\mathcal{O}(-1) := \mathcal{O}_{\mathbb{C}^{n+1}}(-1) := \{(z,\ell) \in \mathbb{C}^{n+1} \times \mathbb{P}_n \ ; \ z \in \ell\}.$$

Let us verify that it is a holomorphic line bundle. In the chart

$$U_o = \{[1, z] \ ; \ z \in \mathbb{C}^n\}$$

we have the holomorphic section

$$\mathbf{e}_o([1, z]) = ((1, z), [1, z]) \subset \mathbb{P}_n$$

which defines a frame for $\pi : \mathcal{O}(-1) \to \mathbb{P}_n$ over U_o. More generally, in the chart

$$U_j = \{[z^1, ..., z^j, 1, z^{j+1}, ..., z^n] \ ; \ z \in \mathbb{C}^n\}$$

we have the section

$$\mathbf{e}_j([z^1, ..., z^j, 1, z^{j+1}, ..., z^n]) = ((z^1, ..., z^j, 1, z^{j+1}, ..., z^n), [z^1, ..., z^j, 1, z^{j+1}, ..., z^n]),$$

which is obviously holomorphic. Thus $\mathcal{O}(-1) \to \mathbb{P}_n$ is a holomorphic line bundle. (One can also easily check that the change of frame over $U_i \cap U_j$ is holomorphic.) The holomorphic line bundle $\mathcal{O}(-1) \to \mathbb{P}_n$ is called the *tautological line bundle*, and sometimes also *the universal line bundle*.

The line bundle $\mathcal{O}(-1) \to \mathbb{P}_n$ is not trivial. We shall see this non-triviality by showing that the dual bundle is not trivial.

The dual line bundle to $\mathcal{O}(-1) \to \mathbb{P}_n$ is called the *hyperplane line bundle*, and denoted $\mathcal{O}(1) \to \mathbb{P}_n$. Its fiber $\mathcal{O}(1)_\ell$ consists of the set of linear functionals on the 1-dimensional subspace ℓ of $\mathbb{C}^{n+1}$. Thus a linear function $\lambda : \mathbb{C}^{n+1} \to \mathbb{C}$ defines a section $s_\lambda : \mathbb{P}_n \to \mathcal{O}(1)$ of $\mathcal{O}(1) \to \mathbb{P}_{n+1}$ via the formula

$$\langle s_\lambda(\ell), (z, \ell)\rangle := \lambda(z).$$

It is easy to check that this section is holomorphic, and that $s_\lambda(\ell) = 0$ if and only if $\ell \subset \text{Kernel}(\lambda)$. (As we shall see later on, every holomorphic section of $\mathcal{O}(1) \to \mathbb{P}_n$ is of the form s_λ for some $\lambda \in (\mathbb{C}^{n+1})^*$.)

By contrast, $\mathcal{O}(-1) \to \mathbb{P}_n$ has no global holomorphic sections other than the zero section. Indeed, suppose $\mathcal{O}(-1) \to \mathbb{P}_n$ has a holomorphic section $\theta : \mathbb{P}_n \to \mathcal{O}(-1)$. Given any linear functional $\lambda \in (\mathbb{C}^{n+1})^* - \{0\}$, we have the section s_λ. The duality pairing

$$g(\ell) := \langle s_\lambda(\ell), \theta(\ell)\rangle$$

is thus a well-defined holomorphic function on $\mathbb{P}_n$. Since $\mathbb{P}_n$ is compact, g must be constant. Since every non-identically zero linear functional on $\mathbb{C}^{n+1}$ has a non-trivial kernel for $n \geq 1$, $g = 0$, and therefore θ vanishes on the complement of the zero locus of s_λ. But by the identity principle, $\theta = 0$. In particular, $\mathcal{O}(-1) \to \mathbb{P}_n$ *is not trivial*, and therefore neither is $\mathcal{O}(1) \to \mathbb{P}_n$.

In the sequel we shall often see the line bundles $\mathcal{O}(-1) \to \mathbb{P}_n$ and $\mathcal{O}(1) \to \mathbb{P}_n$, as well as their integer tensor products $\mathcal{O}(k)$, defined by

$$\mathcal{O}(k) := \mathcal{O}\left(\frac{k}{|k|}\right)^{\otimes |k|}$$

for all $k \in \mathbb{Z}$. ◇

An *exact sequence of holomorphic vector bundles* is a collection of holomorphic vector bundles $E_i \to X$, $i = 1, 2, ...$ together with holomorphic vector bundle maps $F_i : E_i \to E_{i+1}$ such that

$$\mathrm{Image}(F_i) = \mathrm{Kernel}(F_{i+1})$$

for all i. A *short exact sequence of holomorphic vector bundles* is an exact sequence of the form

$$0 \mapsto S \xrightarrow{\iota} E \xrightarrow{q} Q \to 0.$$

By definition of exact sequence the maps ι and q are respectively injective and surjective. Thus we can identify S with a holomorphic subbundle of E, while Q is the vector bundle whose fiber over $x \in X$ is the quotient space E_x/S_x.

1.2.6 Example (The tautological quotient bundle). By construction, the tautological line bundle $\mathcal{O}(-1) \to \mathbb{P}_n$ is a subbundle of the trivial vector bundle $\mathbb{C}^{n+1} \times \mathbb{P}_n$ of rank $n + 1$ over $\mathbb{P}_n$. We can therefore define a holomorphic vector bundle $\mathbf{Q} \to \mathbb{P}_n$ by the short exact sequence

$$0 \to \mathcal{O}(-1) \to \mathbb{C}^{n+1} \times \mathbb{P}_n \to \mathbf{Q} \to 0;$$

the second arrow is just the natural inclusion, and the last arrow is the map defined over $\ell \in \mathbb{P}_n$ by the canonical projection $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}/\ell$. The fibers of $\mathbf{Q}$ are therefore of complex dimension n, but the vector bundle $\mathbf{Q} \to \mathbb{P}_n$ is not trivial.

The vector bundle $\mathbf{Q} \to \mathbb{P}_n$ is called the *tautological quotient bundle*. $\diamond$

1.3 Infinitesimal Complex Structure

Almost complex manifolds

1.3.1 Definition. An *almost complex structure* on a manifold M is a section J of the vector bundle $\mathrm{Hom}(T_M, T_M)$, such that $J^2 = -I$. A pair (M, J) of a manifold and an almost complex structure is called an *almost complex manifold.*

1.3.2 Proposition. *The dimension of an almost complex manifold is even.*

Proof. Suppose (M, J) is an almost complex manifold. Then for any $p \in M$, the linear map $J : T_{M,p} \to T_{M,p}$ satisfies

$$0 < (\det J)^2 = \det(J^2) = \det(-I) = (-1)^{\dim(M)}.$$

(Here $\det : \mathscr{L}(T_{M,p}, T_{M,p}) \to \mathbb{R}$ is the function defined on linear transformations of $T_{M,p}$ by the equation

$$\det(A) v_1 \wedge ... \wedge v_n = (Av_1) \wedge ... \wedge (Av_n),$$

where $v_1, ..., v_n$ is any basis of $T_{M,p}$.) Thus $\dim_{\mathbb{R}}(M)$ is even. $\square$

The splitting

The endomorphism J acts as multiplication by $\sqrt{\text{-}1}$; an idea we now make more precise.

The minimal polynomial of an almost complex structure J is

$$P(z) = z^2 + 1 = (z + \sqrt{\text{-}1})(z - \sqrt{\text{-}1}).$$

Thus J has no eigenvectors in the real vector space $T_{M,x}$. However, the natural extension of J to the complexified tangent space $T_{M,x} \otimes \mathbb{C} = T_{M,x} \otimes_{\mathbb{R}} \mathbb{C}$, i.e., $J(c \otimes v) := c \otimes Jv$ for all $c \in \mathbb{C}$ and all $v \in T_{M,x}$ is diagonalizable. Moreover, since J is real, if $\xi \in T_{M,x} \otimes \mathbb{C}$ is an eigenvector with eigenvalue λ then

$$J\bar{\xi} = \overline{J\xi} = \overline{\lambda\xi} = \bar{\lambda}\bar{\xi}.$$

Writing

$$T_M^{1,0} := \left\{ v \in T_M \otimes \mathbb{C} \;;\; Jv = \sqrt{\text{-}1} v \right\} \quad \text{and} \quad T_M^{0,1} := \left\{ v \in T_M \otimes \mathbb{C} \;;\; Jv = -\sqrt{\text{-}1} v \right\},$$

we obtain a splitting

$$T_M \otimes \mathbb{C} = T_M^{1,0} \oplus T_M^{0,1},$$

and that the involution given by complex conjugation interchanges $T_M^{1,0}$ and $T_M^{0,1}$.

Consider the composite map

$$s^{1,0} : T_M \hookrightarrow T_M \otimes \mathbb{C} = T_M^{1,0} \oplus T_M^{0,1} \to T_M^{1,0},$$

where the right-most arrow is the projection onto the first factor. For $v \in T_M^{1,0}$ we have $v = v^{1,0} + v^{0,1}$ where $Jv^{1,0} = \sqrt{\text{-}1}v^{1,0}$ and $Jv^{0,1} = -\sqrt{\text{-}1}v^{0,1}$. Since $v = \bar{v}$, $v^{0,1} = \overline{v^{1,0}}$. Thus $s^{1,0} : v \mapsto v^{1,0}$ is a bijection satisfying

$$2\text{Re } s^{1,0}v = v \quad \text{and} \quad s^{1,0}Jv = \sqrt{\text{-}1}v.$$

In words, $s^{1,0}$ is a real isomorphism (its inverse is 2Re) that intertwines J and $\sqrt{\text{-}1}$.

Remark. Since $Jv = \sqrt{\text{-}1}(v^{1,0} - v^{0,1})$ we have $s^{1,0}v = v^{1,0} = \frac{1}{2}(v - \sqrt{\text{-}1}Jv)$. ⋄

Differential (p, q)-forms

Let (M, J) be an almost complex manifold. The splitting $T_M \otimes \mathbb{C} = T_M^{1,0} \oplus T_M^{0,1}$ induces the splitting $T_M^* \otimes \mathbb{C} = T_M^{*1,0} \oplus T_M^{*0,1}$ (which is the same as the splitting induced by the map $J^* : T_{M,p}^* \to T_{M,p}^*$ defined by $(J^*\alpha)v = \alpha(Jv)$ for all $\alpha \in T_{M,p}^*$ and all $v \in T_{M,p}$), and we therefore obtain the splittings

$$\Lambda^r(T_M^* \otimes \mathbb{C}) = \bigoplus_{p+q=r} \Lambda^{p,q}(T_M^*), \tag{1.1}$$

where

$$\Lambda^{p,q}(T_M^*) := \underbrace{T_M^{*1,0} \bigwedge ... \bigwedge T_M^{*1,0}}_{p \text{ copies}} \bigwedge \underbrace{T_M^{*0,1} \bigwedge ... \bigwedge T_M^{*0,1}}_{q \text{ copies}}.$$

The sections of $\Lambda^{p,q}(T_M^*)$ are called differential forms of bi-degree (p, q), or simply (p, q)-forms. If $T_M^{*1,0}$ admits a local frame $\alpha_1, ..., \alpha_n$ of complex-valued 1-forms, then the forms

$$\alpha^I \wedge \bar{\alpha}^J := \alpha^{i_1} \wedge ... \wedge \alpha^{i_p} \wedge \bar{\alpha}^{j_1} \wedge ... \wedge \bar{\alpha}^{j_q}, \tag{1.2}$$

where $I = (i_1, ..., i_p) \in \{1, ..., n\}^p$ and $J = (j_1, ..., j_q) \in \{1, ..., n\}^q$, provide a local basis of sections for $\Lambda^{p,q}(T_M^*)$. In particular, one can write any (p, q)-form θ as

$$\theta = \theta_{I\bar{J}}\alpha^I \wedge \bar{\alpha}^J,$$

for some collection of functions $\theta_{I\bar{J}}$. And if the collection of functions is skew symmetric with respect to the rearrangement of the indices $i_1, ..., i_p$ and $j_1, ..., j_q$ then the decomposition is unique.

The exterior operators

We define the operators ∂ and $\bar{\partial}$ on functions, and taking values in local sections of the complexified cotangent bundle, by

$$\partial = \Pi^{1,0} \circ d \qquad \text{and} \qquad \bar{\partial} := \Pi^{0,1} \circ d,$$

where $\Pi^{1,0}$ and $\Pi^{0,1}$ are the projections from $T_M^* \otimes \mathbb{C}$ to $T_M^{*1,0}$ and $T_M^{*0,1}$ respectively, and by d we actually mean the $\mathbb{C}$-linear extension of d to $\mathbb{C}$-valued functions. Similarly, one has projections $\Pi^{p,q} : \Lambda^{p+q}(T_M^* \otimes \mathbb{C}) \to \Lambda^{p,q}(T_M^*)$, and we may define ∂ and $\bar{\partial}$ on (p,q)-forms respectively by the formulas

$$\partial = \Pi^{p+1,q} \circ d \quad \text{and} \quad \bar{\partial} = \Pi^{p,q+1} \circ d.$$

We can extend ∂ and $\bar{\partial}$ to general complexified forms by

$$\partial \sum_{p,q} \theta^{p,q} = \sum_{p,q} \partial \theta^{p,q} \quad \text{and} \quad \bar{\partial} \sum_{p,q} \theta^{p,q} = \sum_{p,q} \bar{\partial} \theta^{p,q}.$$

If α is a (p,q)-form then $d\alpha$ is a $p+q+1$-form, so a priori

$$d\alpha = (d\alpha)^{p+q+1,0} + (d\alpha)^{p+q,1} + \cdots + (d\alpha)^{0,p+q+1}.$$

But in fact at most 4 of these $p+q+2$ terms are non-zero.

1.3.3 PROPOSITION. *If α is a (p,q)-form then*

$$d\alpha = (d\alpha)^{p+2,q-1} + \partial\alpha + \bar{\partial}\alpha + (d\alpha)^{p-1,q+2}.$$

Proof. If α is a $(1,0)$-form the result holds by the decomposition (1.1) (and the first term is zero). By conjugation the result also holds for $(0,1)$-forms, and hence by linearity the result holds for 1-forms. Since all forms are linear combinations of wedge-products of 1-forms, the proposition follows from the Leibniz rule by induction. $\square$

Almost complex structure for a complex manifold

A complex manifold X always has an almost complex structure J defined as follows. Let $(z^1, ..., z^n)$ be holomorphic coordinates on a neighborhood of some point p, and let $x^i = \text{Re } z^i$ and $y^i = \text{Im } z^i$. We define J by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i} \qquad \text{and} \qquad J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}. \tag{1.3}$$

If $w = u + \sqrt{\text{-}1}v$ is another holomorphic coordinate system in some neighborhood of p, then the functions z^i depend holomorphically on w, which is to say,

$$\frac{\partial}{\partial u^i} = \frac{\partial x^k}{\partial u^i}\frac{\partial}{\partial x^k} + \frac{\partial y^k}{\partial u^i}\frac{\partial}{\partial y^k} = \frac{\partial y^k}{\partial v^i}\frac{\partial}{\partial x^k} - \frac{\partial x^k}{\partial v^i}\frac{\partial}{\partial y^k}$$

and

$$\frac{\partial}{\partial v^i} = \frac{\partial x^k}{\partial v^i}\frac{\partial}{\partial x^k} + \frac{\partial y^k}{\partial v^i}\frac{\partial}{\partial y^k} = -\frac{\partial y^k}{\partial u^i}\frac{\partial}{\partial x^k} + \frac{\partial x^k}{\partial u^i}\frac{\partial}{\partial y^k}$$

we compute that

$$J\left(\frac{\partial}{\partial u^i}\right) = J\left(\frac{\partial y^k}{\partial v^i}\frac{\partial}{\partial x^k} - \frac{\partial x^k}{\partial v^i}\frac{\partial}{\partial y^k}\right) = \frac{\partial y^k}{\partial v^i}\frac{\partial}{\partial y^k} + \frac{\partial x^k}{\partial v^i}\frac{\partial}{\partial x^k} = \frac{\partial}{\partial v^i}$$

and

$$J\left(\frac{\partial}{\partial v^i}\right) = J\left(-\frac{\partial y^k}{\partial u^i}\frac{\partial}{\partial x^k} + \frac{\partial x^k}{\partial u^i}\frac{\partial}{\partial y^k}\right) = -\frac{\partial y^k}{\partial u^i}\frac{\partial}{\partial y^k} - \frac{\partial x^k}{\partial u^i}\frac{\partial}{\partial x^k} = -\frac{\partial}{\partial u^i},$$

which shows that J is well-defined.

A local basis for sections of the space $T_X^{1,0}$ is then given by the vectors

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{\text{-}1}\frac{\partial}{\partial y^i}\right), \quad 1 \le i \le n,$$

and one for $T_X^{0,1}$ by

$$\frac{\partial}{\partial \bar{z}^i} = \overline{\left(\frac{\partial}{\partial z^i}\right)} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{\text{-}1}\frac{\partial}{\partial y^i}\right), \quad 1 \le i \le n.$$

Next we compute transition functions for $T_X^{1,0}$ in the case where X is a complex manifold. If X has a covering by local holomorphic coordinate neighborhoods, two of which are U_1 and U_2 with coordinates $z_1 = (z_1^1, ..., z_1^n)$ and $z_2 = (z_2^1, ..., z_2^n)$, then $T_X^{1,0}$ is locally trivial on both U_1 and U_2, using as local bases the complex vector fields

$$\frac{\partial}{\partial z_j^i},\ i = 1, ..., n,\ \text{on } U_j,\ j = 1, 2.$$

Assuming these two coordinate charts intersect, the transition functions may be computed using the chain rule. We have

$$\frac{\partial}{\partial z_2^j} = \frac{\partial z_1^i}{\partial z_2^j}\frac{\partial}{\partial z_1^i}.$$

Thus if $z_2 = F(z_1)$, the transition functions are given by the inverse-transpose of the matrix ∂F, which coefficients are holomorphic functions. Since $T_X^{0,1} = \overline{T_X^{1,0}}$, the transition functions for $T_X^{0,1}$ are $\overline{\partial F}$ with respect the basis of complex vector fields

$$\frac{\partial}{\partial \bar{z}_j^i},\ i = 1, ..., n,\ \text{on } U_j,\ j = 1, 2,$$

and thus are not given by a matrix of holomorphic functions in this basis.

As for the cotangent bundle, the sections $dz^1, ..., dz^n$ and $d\bar{z}^1, ..., d\bar{z}^n$ form the frames for $T_X^{*1,0}$ and $T_X^{*0,1}$ dual to $\partial_{z^1}, ..., \partial_{z^n}$ and $\partial_{\bar{z}^1}, ..., \partial_{\bar{z}^n}$ respectively. A general (p, q)-form then locally has the form

$$\alpha = \alpha_{I\bar{J}} dz^I \wedge d\bar{z}^J.$$

One can compute directly that a change of local coordinates defines transition functions for $T_X^{*1,0}$ and $T_X^{*0,1}$ that are the transposes of the inverses of the transition functions for $T_X^{1,0}$ and $T_X^{0,1}$ respectively. Using the transformation laws for the spaces $T_X^{*1,0}$ and $T_X^{*0,1}$, together with the fact that the sections (1.2) provide local trivializations for $\Lambda^{p,q}(T_X^*)$, one can easily compute the transition functions for $\Lambda^{p,q}(T_X^*)$.

Real submanifolds of complex manifolds

1.3.4 PROPOSITION. *Let X be a complex manifold and let $S \subset X$ be a real submanifold such that $J\xi \in T_S$ for any $\xi \in T_S$. Then S is a complex submanifold of X.*

Proof. Since $J_S := J|_{T_S}$ is injective and maps T_S to itself, J_S is an almost complex structure for S, and thus S is even dimensional, say $\dim_{\mathbb{R}}(S) = 2k$. Let us choose holomorphic coordinates $z^1 = x^1 + \sqrt{\text{-}1}y^1, ..., z^n = x^n + \sqrt{\text{-}1}y^n$ on an open subset of X containing a point p, in such a way that for each $1 \le i \le k$, either $\frac{\partial}{\partial x^i}$ or $\frac{\partial}{\partial y^i}$ is tangent to S at the point p. Since $J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}$ and $J\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}$, our hypothesis on S implies that $\frac{\partial}{\partial x^i}$ is tangent to S if and only if $\frac{\partial}{\partial y^i}$ is tangent to S.

By the Implicit Function Theorem $(\text{Re } z^1, \text{Im } z^1, ..., \text{Re } z^k, \text{Im } z^k)$ restrict to local coordinates on S. Moreover, since S is a smooth submanifold of X, there exist smooth $\mathbb{C}$-valued functions $f^1, ..., f^{n-k}$ of $2n - 2k$ variables such that a point lies in S if and only if its coordinates in the chart $z = (z^1, ..., z^n)$ satisfy the set of $n - k$ equations

$$z^{k+i} = f^i(z^1, \bar{z}^1, ..., z^k, \bar{z}^k), \quad 1 \le i \le n-k.$$

By the Chain Rule,

$$dz^{k+i} = \sum_{j=1}^{k} \frac{\partial f^i}{\partial z^j} dz^j + \sum_{j=1}^{k} \frac{\partial f^i}{\partial \bar{z}^j} d\bar{z}^j.$$

Since $dz^{k+i}\left(\frac{\partial}{\partial \bar{z}^j}\right) = 0$ for all $i \ge 1$ and $j \le k$, we have

$$\frac{\partial f^i}{\partial \bar{z}^j} = dz^{k+i}\left(\frac{\partial}{\partial \bar{z}^j}\right) = 0,$$

i.e., the functions f^i are holomorphic on S. It follows that S is the graph of the holomorphic map $(f^1, ..., f^{n-k}) : U \subset \mathbb{C}^k \to \mathbb{C}^{n-k}$ in the coordinate chart in question, so S is a complex submanifold of X. □

Integrability of almost complex structures

Given a real manifold with an almost complex structure, it is natural to inquire whether the manifold in question is a complex manifold. More precisely, can we find a complex atlas such that the almost complex structure in question is associated to the complex atlas in the manner described in the previous paragraph.

1.3.5 DEFINITION. An almost complex structure that can be obtained from an underlying complex structure is said to be integrable.

1.3.6 PROPOSITION. *Let $\Omega \subset \mathbb{R}^{2n}$ be an open set, and let J be an almost complex structure on Ω. Suppose we have open sets $\Omega_1, \Omega_2 \subset \Omega$ and diffeomorphisms $F_1 : \Omega_1 \to U_1 \subset \mathbb{R}^{2n}$ and $F_2 : \Omega_2 \to U_2 \subset \mathbb{R}^{2n}$ such that the almost complex structures*

$$I_i := dF_i J d(F_i^{-1}), \quad i = 1, 2$$

are constant linear transformations. Then there there is are matrices $A_1, A_2 \in GL(2n, \mathbb{R})$ such that the coordinate maps $A_1 \circ F_1$ and $A_2 \circ F_2$ have holomorphic transition functions.

Proof. Observe that $I_i^2 = -\mathrm{Id}$, and thus there exist matrices A_i such that $A_i I_i A_i^{-1} = j$, where j is the standard complex structure given in (1.3). Letting $f_i := A_i F_i$, we have

$$df_1 J = j df_1 \quad \text{and} \quad df_2 J = j df_2.$$

Thus, with $\Phi := f_2 \circ f_1^{-1} : A_1(U_1) \to A_2(U_2)$,

$$d\Phi \circ j = df_2 d(f_1^{-1}) j = df_2 J df_1^{-1} = j df_2 d(f_1^{-1}) = j d\Phi.$$

Thus Φ is holomorphic, as desired. □

1.3.7 Definition. Let J be an almost complex structure on a manifold M. Define the Nijenhuis tensor $N_J : T_M \otimes T_M \to T_M$ by

$$N_J(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY].$$

If $N_J \equiv 0$, we say that J is involutive.

Observe that if f is a function on M and X and Y are local vector fields, then

$$\begin{aligned} N_J(fX,Y) &= [fX,Y] + J([fJX,Y] + [fX,JY]) - [fJX,JY] \\ &= f[X,Y] - (Yf)X + J(f[JX,Y] - (Yf)JX + f[X,Y] - J(Yf)X) \\ &\qquad -f[JX,JY] + J(Yf)JX \\ &= fN_J(X,Y). \end{aligned}$$

Since $N_J(X,Y) = -N_J(Y,X)$, we see that $N_J(X,Y)$ depends only on the pointwise value of X and Y, which shows that N_J is a tensor, i.e., a section of $T_M \otimes T_M^* \otimes T_M^*$. We remark also that N_J is bilinear, so it can be extended to the complexification. With the extension still denoted N_J, if $X, Y \in \Gamma(M, T_M^{1,0})$ then

$$N_J(X,Y) = 2\sqrt{\text{-}1}(J[X,Y] - \sqrt{\text{-}1}[X,Y]). \tag{1.4}$$

The notion of involutive almost complex structure can be expressed in four other ways, as the next proposition shows.

1.3.8 Proposition. *Let J be an almost complex structure on a manifold M. Then the following are equivalent.*

1. $d = \partial_J + \bar{\partial}_J$ *on all 1-forms.*
2. $d = \partial_J + \bar{\partial}_J$ *on all forms.*
3. $\bar{\partial}_J \circ \bar{\partial}_J = 0$.

4. $[\Gamma(M, T_M^{1,0}), \Gamma(M, T_M^{1,0})] \subset \Gamma(M, T_M^{1,0})$.

5. $N_J \equiv 0$.

Proof. We will show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1 \iff 4 \iff 5$.

1 ⇒ 2 : Since $\Pi^{1,0} + \Pi^{0,1} = \mathrm{Id}$, $dh = \partial_J h + \bar{\partial}_J h$ for functions h. Since all forms are wedge products of 1-forms, the Leibniz rule determines d its action on 1-forms.

2 ⇒ 3 : If 2 is assumed then

$$0 = d^2 = \partial_J^2 + (\partial_J \bar{\partial}_J + \bar{\partial}_J \partial_J) + \bar{\partial}_J^2.$$

and thus we have 3.

3 ⇒ 1 : Suppose 3 holds. Since every 1-form is a sum of 1-forms of the form fdg, it suffices to show that $d(fdg) = \partial_J(fdg) + \bar{\partial}_J(fdg)$. Since $dh = \partial_J h + \bar{\partial}_J h$ for functions h, $\partial_J dg + \bar{\partial}_J dg = (\partial_J \bar{\partial}_J + \bar{\partial}_J \partial_J)g$, and hence

$$\partial_J(fdg) + \bar{\partial}_J(fdg) = df \wedge dg + f(\partial_J \bar{\partial}_J + \bar{\partial}_J \partial_J)g = d(fdg) + (\partial_J \bar{\partial}_J + \bar{\partial}_J \partial_J)g.$$

It remains only to show that $(\partial_J \bar{\partial}_J + \bar{\partial}_J \partial_J)g = 0$. But by Proposition 1.3.3

$$\begin{aligned} 0 &= d^2 g \\ &= d(\partial_J g) + d(\bar{\partial}_J g) \\ &= \left[\partial_J^2 g + \Pi^{2,0} d\bar{\partial}_J g\right] + \left[(\partial_J \bar{\partial}_J + \bar{\partial}_J \partial_J)g\right] + \left[\bar{\partial}_J^2 g + \Pi^{0,2} d\partial_J g\right], \end{aligned}$$

so $(\partial_J \bar{\partial}_J + \bar{\partial}_J \partial_J)g = 0$ for all functions g and *any* almost complex structure (not necessarily satisfying $\bar{\partial}_J^2 = 0$). Hence 1 holds.

1 ⟺ 4 : By complex conjugation 1 holds if and only if it holds for $(1,0)$-forms. For any $(1,0)$-form α 1 is equivalent to the statement that for any $X, Y \in \Gamma(M, T_M^{1,0})$,

$$d\alpha(\overline{X}, \overline{Y}) = 0. \tag{1.5}$$

By Cartan's formula for the exterior derivative, we have

$$d\alpha(\overline{X}, \overline{Y}) = \overline{X}\alpha(\overline{Y}) - \overline{Y}\alpha(\overline{X}) - \alpha([\overline{X}, \overline{Y}]) = -\alpha([\overline{X}, \overline{Y}]),$$

which shows that $1 \Rightarrow 4$. Conversely, if 4 holds then $\overline{[X, Y]} = [\overline{X}, \overline{Y}]$ is annihilated by all $(1,0)$-forms, so $[X, Y]$ is a $(1,0)$-vector field. Thus 1 holds.

4 ⟺ 5 : Let $X, Y \in \Gamma(M, T_M^{1,0})$. If $N_J \equiv 0$ then by (1.4), $J[X, Y] = \sqrt{\text{-}1}[X, Y]$, so that $[X, Y] \in \Gamma(M, T_M^{1,0})$. Conversely,if $[\Gamma(M, T_M^{1,0}), \Gamma(M, T_M^{1,0})] \subset \Gamma(M, T_M^{1,0})$ then by (1.4), $N_J(X, Y) = 0$, while

$$N_J(X, \overline{Y}) = [X, \overline{Y}] + \sqrt{\text{-}1}J([X, \overline{Y}] - [X, \overline{Y}]) - [X, \overline{Y}] = 0.$$

Since N_J is bilinear and antisymmetric, $N_J \equiv 0$. □

1.3.9 PROPOSITION. *If J is integrable then J is involutive, i.e., $N_J = 0$.*

Proof. By Proposition 1.3.6 we can assume that J is the standard almost complex structure. Then we know that $\bar{\partial}\bar{\partial} = 0$, and the result follows from Proposition 1.3.8. □

The main result of integrability theory is the following theorem.

1.3.10 THEOREM (Newlander-Nirenberg Theoerm). *Let (M, J) be an almost complex manifold and let $\bar{\partial}_J$ be the associated 'Cauchy-Riemann' exterior operators. Then J is integrable if and only if*

$$\bar{\partial}_J \circ \bar{\partial}_J = 0.$$

One direction— the easy direction— is just Propositions 1.3.9. The converse is the difficult part, which we will prove at the end of Chapter 2 after we have developed the techniques necessary for our proof.

1.3.11 REMARK. If the manifold M and the almost complex structure J are real analytic, Theorem 1.3.10 can be seen to follow from the holomorphic version of Frobenius's Integrability Theorem. However, in the general smooth case a proof of Theorem 1.3.10 requires more work. ⋄

The $\bar{\partial}$ operator for a holomorphic vector bundle

Let X be a complex manifold and let $E \to X$ be a holomorphic vector bundle. One can define a homogeneous first order differential operator on sections of E as follows. If $e_1, ..., e_r$ is a holomorphic frame, then this operator is given by the formula

$$s = s^i e_i \mapsto (\bar{\partial} s^i) \otimes e_i. \tag{1.6}$$

Note that if $\tilde{e}_j$ is another holomorphic frame then there is an invertible matrix of holomorphic functions (g^i_j) such that

$$\tilde{e}_j = g^i_j e_i \quad \text{and thus} \quad s^j = \tilde{s}^i g^j_i.$$

We therefore obtain

$$(\bar{\partial}\tilde{s}^i) \otimes \tilde{e}_i = \bar{\partial}(\tilde{s}^j) \otimes g^i_j e_i = \bar{\partial}(\tilde{s}^j g^i_j) \otimes \tilde{e}_i = (\bar{\partial} s^i) \otimes e_i,$$

which shows that (1.6) is independent of the choice of frame.

1.3.12 DEFINITION. The differential operator given in (1.6) is denoted $\bar{\partial}$. ⋄

The $\bar{\partial}$-operator just defined acts on sections of the holomorphic vector bundle $E \to X$, and more generally on E-valued (p, q)-forms via the same formula. If we continue to denote the resulting operator by $\bar{\partial}$ then one sees that

$$\bar{\partial}\bar{\partial} = 0 \quad \text{and} \quad \bar{\partial}(\alpha \otimes s) = \bar{\partial}\alpha \otimes s + (-1)^{\deg \alpha} \alpha \wedge \bar{\partial} s. \tag{1.7}$$

In fact, Equation (1.7) characterizes holomorphic vector bundles, as the next theorem shows.

1.3.13 Theorem. *Let (X, J) be an almost complex manifold and let $E \to X$ be a smooth vector bundle. Assume there exist operators $D'' : \Gamma(X, \Lambda_X^{p,q} \otimes E) \to \Gamma(X, \Lambda_X^{p,q+1} \otimes E)$ such that*

$$D''D'' = 0 \quad \textit{and} \quad D''\alpha \otimes s = \bar{\partial}_J \alpha \otimes s + (-1)^{\deg \alpha} \alpha \wedge D''s$$

for all smooth sections s of $E \to X$ and all smooth (p, q)-forms α on X. Then

(i) *the almost complex structure J is integrable, and*

(ii) *there is a unique holomorphic vector bundle $\mathcal{E} \to X$ and a smooth isomorphism of vector bundles $\Phi : E \to \mathcal{E}$ such that*

$$\bar{\partial}\Phi(s) = \Phi(D''s)$$

for all smooth sections s of E.

Theorem 1.3.13 is similar to Theorem 1.3.10, and is also proved at the end of Chapter 2.

1.4 Hermitian metrics

The terminology 'Hermitian metrics' is ambiguous in complex geometry: on the one hand, it can refer to metrics for a complex vector bundle, and on the other, for certain Riemannian metrics on almost complex manifolds. To avoid confusion when it arises, we refer to the first type of metric as a Hermitian metric, and to the second type of metric as a Hermitian Riemannian metric.

1.4.1 Hermitian metrics for complex vector bundles

Let M be a manifold and let $E \to M$ be a complex vector bundle.

1.4.1 Definition. A Hermitian metric for $E \to M$ is a section $\mathfrak{H}$ of the bundle $E^* \otimes \overline{E^*} \to M$ such that for all $x \in M$ and $v, w \in E_x$,

(i) $\langle \mathfrak{H}, v \otimes \bar{w} \rangle = \overline{\langle \mathfrak{H}, w \otimes \bar{v} \rangle}$ (i.e., $\mathfrak{H}$ is Hermitian symmetric), and

(ii) $\langle \mathfrak{H}, v \otimes \bar{v} \rangle > 0$ for all $v \neq 0$ (i.e., $\mathfrak{H}$ is positive-definite).

In other words, $\mathfrak{H}$ defines a sesquilinear, positive definite Hermitian form on each fiber E_x of the vector bundle $E \to M$. ◇

A Hermitian metric is often thought of as a smooth family of sesquilinear forms h on $E \to M$, via the identification

$$h(v, w) := \langle \mathfrak{H}, v \otimes \bar{w} \rangle, \quad v, w \in E_x.$$

We shall not be too fussy on which notation we use. Most of the time we think of a metric as a sesquilinear form, but from time to time it will be convenient to see a Hermitian metric as a section of a bundle

If $\alpha^1, ..., \alpha^r$ is a frame for E^* over an open set U then one can write

$$\mathfrak{H} = \mathfrak{H}_{i\bar{j}} \alpha^i \otimes \bar{\alpha}^j$$

for functions $\mathfrak{H}_{i\bar{j}}$ over U satisfying

$$\overline{\mathfrak{H}_{i\bar{j}}} = \mathfrak{H}_{j\bar{i}} \quad \text{and} \quad \mathfrak{H}_{i\bar{j}} a^i \bar{a}^j > 0 \text{ for all } a \in \mathbb{C}^r - \{0\}.$$

That is to say, at each $x \in U$ the matrix $\left(\mathfrak{H}_{i\bar{j}}(x)\right)_{i,j=1}^r$ is Hermitian and positive-definite. If the frame $\alpha^1, ..., \alpha^r$ is smooth then the regularity of the functions $\mathfrak{H}_{i\bar{j}}$ is declared to be the regularity of $\mathfrak{H}$.

1.4.2 Hermitian Riemannian metrics

Let (M, J) be an almost complex manifold.

1.4.2 DEFINITION. A Riemannian metric g on M is said to be Hermitian if $J^* g = g$, i.e.,

$$g(J\xi, J\eta) = g(\xi, \eta)$$

for all $x \in M$ and all $\xi, \eta \in T_{M,x}$. $\diamond$

The condition of J-invariance puts some strong restrictions on the form of the metric; restrictions that are best seen in the decomposition $T_M \otimes \mathbb{C} = T_M^{1,0} \oplus T_M^{0,1}$ of the complexified tangent space. In terms of the splitting $T_M \otimes \mathbb{C} = T_M^{1,0} \oplus T_M^{0,1}$, write (the complexification of) the metric as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

i.e.,

$$g(v_1 + \bar{w}_1, v_2 + \bar{w}_2) = v_1 \cdot A v_2 + v_1 \cdot B \bar{w}_2 + \bar{w}_1 \cdot C v_2 + \bar{w}_1 \cdot D \bar{w}_2, \quad v_i, w_i \in T_M^{1,0}.$$

We shall now analyze the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$.

First, for $v_1, v_2 \in T_M^{1,0}$ one has

$$g(v_1, v_2) = g(Jv_1, Jv_2) = g(\sqrt{\text{-}1} v_1, \sqrt{\text{-}1} v_2) = -g(v_1, v_2),$$

and thus $A = 0$. Since g comes from a real metric, $g(\bar{v}, \bar{w}) = \overline{g(v, w)}$, and therefore $D = 0$.

Again by conjugation we have $\overline{g(v, \overline{w})} = g(\overline{v}, w)$, and thus

$$\overline{B} = C.$$

Finally, since g is symmetric, $C = B^{\text{trans}}$, and therefore

$$B^\dagger = B,$$

where † means transpose conjugate. Thus we have

$$g = \begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix}$$

for some Hermitian metric h for $T_M^{1,0}$. We therefore see that Hermitian Riemannian metrics for M are in one-to-one correspondence with Hermitian metrics for $T_M^{1,0}$. This identification partly explains the aforementioned ambiguity in the name ‘Hermitian metric’.

1.4.3 REMARK. As we shall see, the correspondence between Hermitian Riemannian metrics for T_M and Hermitian metrics for $T_M^{1,0}$ is key for the notion of Kähler metric. $\diamond$

We compute that for real vectors v and w, one has

$$\begin{aligned}2h(s^{1,0}v, \overline{s^{1,0}w}) &= 2g(\tfrac{1}{2}(v - \sqrt{\text{-}1}Jv), \tfrac{1}{2}(w + \sqrt{\text{-}1}Jw)) \\ &= \frac{1}{2}\left(g(v,w) + g(Jv,Jw) + \sqrt{\text{-}1}(g(v,Jw) - g(Jv,w))\right) \\ &= g(v,w) - \sqrt{\text{-}1}g(Jv,w),\end{aligned}$$

where in the last equality we used the Hermitian symmetry of g. Therefore

$$g(v,w) = 2\mathrm{Re}\ h(s^{1,0}v, \overline{s^{1,0}w}).$$

In fact, we can recover the metric g on M from a metric h on $T^{1,0}_M$ using this formula. Indeed, one need only observe that if h is a Hermitian metric for $T^{1,0}_M$ then

$$h(s^{1,0}Jv, \overline{s^{1,0}Jw}) = h(\sqrt{\text{-}1}s^{1,0}v, -\sqrt{\text{-}1}\ \overline{s^{1,0}w}) = h(s^{1,0}v, \overline{s^{1,0}w}),$$

the real (as well as the imaginary) part of $h(s^{1,0}v, \overline{s^{1,0}w})$ is J-invariant.

1.4.4 Definition. The negative of the imaginary part of $2h(s^{1,0}v, \overline{s^{1,0}w})$, namely

$$\omega_g(v,w) = g(Jv,w),$$

is called the *metric form* of the Hermitian Riemannian metric g.

1.4.5 Remark. Observe that if g is a Hermitian Riemannian metric then

$$g(Jv,w) = g(w,Jv) = g(Jw,J^2v) = -g(Jw,v),$$

so that ω_g is a 2-form. The formula $g(v,w) = \omega_g(v,Jw)$ also shows that this 2-form completely determines the Hermitian metric h. ◇

In terms of frames, if $\alpha^1, ..., \alpha^n$ is a local frame for $T^{*1,0}_M$ then a Hermitian metric for $T^{1,0}_M$ may be written

$$h = h_{i\bar{j}}\alpha^i \cdot \bar{\alpha}^j.$$

In terms of this frame, if $g(v,w) := 2\mathrm{Re}\ h(s^{1,0}v, \overline{s^{1,0}w})$ is the associated Hermitian Riemannian metric then

$$\omega_g = \sqrt{\text{-}1}h_{i\bar{j}}\alpha^i \wedge \bar{\alpha}^j.$$

Indeed, the right hand is well-defined because the vector bundles $T^{*1,0}_M \otimes T^{*0,1}_M$ and $T^{*1,0}_M \wedge T^{*0,1}_M$ have the same transition functions, and we compute that

$$\begin{aligned}\sqrt{\text{-}1}h_{i\bar{j}}\alpha^i \wedge \bar{\alpha}^j(v,w) &= \sqrt{\text{-}1}h_{i\bar{j}}\alpha^i \wedge \bar{\alpha}^j\left(s^{1,0}v + \overline{s^{1,0}v}, s^{1,0}w + \overline{s^{1,0}w}\right) \\ &= \sqrt{\text{-}1}h_{i\bar{j}}\left(\left\langle \alpha^i, s^{1,0}v\right\rangle \overline{\langle \alpha^j, s^{1,0}w\rangle} - \left\langle \alpha^i, s^{1,0}w\right\rangle \overline{\langle \alpha^j, s^{1,0}v\rangle}\right) \\ &= -2\mathrm{Im}\ h_{i\bar{j}}\left\langle \alpha^i, s^{1,0}v\right\rangle \overline{\langle \alpha^j, s^{1,0}w\rangle} = -\mathrm{Im}\ 2h(s^{1,0}v, s^{1,0}w),\end{aligned}$$

where the second-to-last equality follows from the Hermitian symmetry of $h_{i\bar{j}}$.

1.4.6 Remark. More commonly, one finds the metric h expressed in terms of a local frame for T_M^* as follows: if $\zeta^1, .., \zeta^n$ is a local frame of 1-forms and we write

$$h = h_{i\bar{j}} s^{1,0}\zeta^i \cdot \overline{s^{1,0}\zeta^j}$$

then, in terms of the dual frame ξ_i of T_M,

$$g_{ij} = 2\text{Re}\, h\left(s^{1,0}\xi_i, s^{1,0}\xi_j\right) = h_{i\bar{j}} + \overline{h_{i\bar{j}}} = h_{i\bar{j}} + h_{j\bar{i}}.$$

Since $h_{i\bar{j}}$ is Hermitian-symmetric,

$$\omega_g = \sqrt{\text{-}1} h_{i\bar{j}} s^{1,0}\zeta^i \wedge \overline{s^{1,0}\zeta^j} = \tfrac{\sqrt{\text{-}1}}{2}\left(h_{i\bar{j}} + h_{j\bar{i}}\right) s^{1,0}\zeta^i \wedge \overline{s^{1,0}\zeta^j} = g_{ij}\tfrac{\sqrt{\text{-}1}}{2} s^{1,0}\zeta^i \wedge \overline{s^{1,0}\zeta^j}.$$

For example, if $z = (z^1, ..., z^n)$ is a local coordinate system on M then, we $z^i = x^i + \sqrt{\text{-}1}y^i$, one can take $\zeta^i = dx^i$. Then $s^{1,0}\zeta^i = dz^i$ and we find that

$$\omega_g = g_{ij}\tfrac{\sqrt{\text{-}1}}{2} dz^i \wedge d\bar{z}^j,$$

a formula more commonly found in texts and articles. ◇

1.4.7 Example. On $M = \mathbb{C}^n$ with coordinates $z = (z^1, ..., z^n)$ and $z^i = x^i + \sqrt{\text{-}1}y^i$, can consider the Euclidean Riemannian metric g_o defined by

$$g_o\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\right) = g_o\left(\tfrac{\partial}{\partial y^i}, \tfrac{\partial}{\partial y^j}\right) = \delta_{ij} \quad \text{and} \quad g_o\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial y^j}\right) = 0, \quad 1 \leq i, j \leq n.$$

Then $g_o = 2\text{Re}\left(\tfrac{\sqrt{\text{-}1}}{2}\delta_{ij} dz^i \cdot d\bar{z}^j\right)$ and $\omega_o := \omega_{g_o} = \tfrac{\sqrt{\text{-}1}}{2}\delta_{ij} dz^i \wedge d\bar{z}^j$. ◇

1.4.3 Volume

The volume form of a Hermitian Riemannian manifold enjoys more regularity than the volume form of general Riemannian manifold.

Riemannian Volume

Given a Riemannian metric g on an m-dimensional real manifold, in a local coordinate system $x = (x^1, ..., x^m)$ one can define the m-form

$$\sqrt{\det\left(g\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\right)\right)} dx^1 \wedge \cdots \wedge dx^m.$$

If one changes to another coordinate system $y = (y^1, ..., y^m)$ then

$$\sqrt{\det\left(g\left(\tfrac{\partial}{\partial y^i}, \tfrac{\partial}{\partial y^j}\right)\right)} dy^1 \wedge \cdots \wedge dy^m = \frac{\left|\det\left(\frac{\partial y^i}{\partial x^j}\right)\right|}{\det\left(\frac{\partial y^i}{\partial x^j}\right)} \sqrt{\det\left(g\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\right)\right)} dx^1 \wedge \cdots \wedge dx^m.$$

If one can choose an atlas whose transition functions all have positive determinant then one gets a globally defined form

$$dV_g := \sqrt{\det\left(g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right)}dx^1 \wedge \cdots \wedge dx^m$$

of top degree with no zeros, i.e., a volume form. The existence of such an atlas is the definition of orientability of a manifold. In particular, if the manifold is complex then the transition functions, being holomorphic, have this property (Proposition **??**). (In this case the manifold is not only orientable, but in fact oriented, i.e., there is a preferred atlas whose transition functions have derivatives with positive determinant.)

Hermitian Volume

If the manifold is complex and the Riemannian metric is furthermore Hermitian then the situation for volume forms is even better: In particular, for a Riemannian Hermitian metric g on a complex manifold X

$$dV_g = \frac{1}{n!}\omega_g^n,$$

where ω_g is the metric form of g. Indeed, if in local complex coordinates $z = (z^1, ..., z^n)$ and corresponding real coordinates $\xi = (\xi^1, ..., \xi^{2n})$ where $\xi^{2i-1} = \text{Re } z^i$, $\xi^{2i} = \text{Im } z^i$, we write $g = g_{ij}d\xi^i d\xi^j$ and $h = h_{\alpha\bar\beta}dz^\alpha \cdot d\bar z^\beta$ then

$$\det\left((g_{ij})_{ij=1}^{2n}\right) = (-1)^n(\det(h_{\alpha\bar\beta}))^2,$$

and thus

$$\begin{aligned}
\omega^n &= \sqrt{\text{-1}}^n h_{\alpha_1\bar\beta_1}...h_{\alpha_n\bar\beta_n}dz^{\alpha_1} \wedge d\bar z^{\beta_1} \wedge ... \wedge dz^{\alpha_1} \wedge d\bar z^{\beta_1}\\
&= \frac{(-1)^{n(n+1)/2}}{(2\sqrt{\text{-1}})^n}h_{\alpha_1\bar\beta_1}...h_{\alpha_n\bar\beta_n}dz^{\alpha_1} \wedge ... \wedge dz^{\alpha_n} \wedge d\bar z^{\beta_1} \wedge ... \wedge d\bar z^{\beta_n}\\
&= \left(\sum_{\alpha_1,...,\bar\beta_n} \text{sgn}\begin{pmatrix} 1 & ... & n & \bar 1 & ... & \bar n\\ \alpha_1 & ... & \alpha_n & \bar\beta_1 & ... & \bar\beta_n\end{pmatrix} h_{\alpha_1\bar\beta_1}...h_{\alpha_n\bar\beta_n}\right)\\
&\qquad\qquad \cdot\frac{(-1)^{n(n+1)/2}}{(2\sqrt{\text{-1}})^n}dz^1 \wedge ... \wedge dz^n \wedge d\bar z^1 \wedge ... \wedge d\bar z^n\\
&= n!\det(h_{\alpha\bar\beta})\frac{dz^1 \wedge d\bar z^1}{2\sqrt{\text{-1}}} \wedge ... \wedge \frac{dz^n \wedge d\bar z^n}{2\sqrt{\text{-1}}}.
\end{aligned}$$

Hermitian Volume on submanifolds

Let us consider a complex submanifold Y of our Hermitian manifold X. We can endow Y with a Hermitian metric simply by restricting the metric from the ambient space.

Now choose local coordinates $z^1, ..., z^n$ on X in such a way that the functions $z^1, ..., z^k$ are coordinates on the submanifold Y. The tangent spaces of Y are then defined by the vanishing of the differentials $dz^{k+1}, ..., dz^n$. In other words, in these coordinates, the Hermitian metric for Y is given by

$$h|_Y = h_{\alpha\bar{\beta}} dz^\alpha \cdot d\bar{z}^\beta,$$

where this time the summation is carried only from 1 to k. In particular, one can carry out all of the above calculations on the submanifold and obtain the following remarkable fact.

1.4.8 THEOREM. *If X is a Hermitian manifold with Hermitian metric form ω and $\iota : Y \hookrightarrow X$ is a k-dimensional submanifold then the associated Hermitian volume form of Y is*

$$dV_{g|_Y} = \frac{1}{k!} \iota^* \omega_g^k = \frac{1}{k!} \omega_{g|_Y}^k.$$

1.5 Connections

1.5.1 Basic definition

Let $\pi : E \to M$ be a vector bundle of rank r. We denote by $\Gamma(M, \mathscr{C}^\infty(E))$ the smooth sections of π.

1.5.1 Definition. A *smooth connection* for a vector bundle $\pi : E \to M$ is linear map

$$D : \Gamma(M, \mathscr{C}^\infty(E)) \to \Gamma(M, \mathscr{C}^\infty(T_M^* \otimes E)),$$

satisfying the Leibniz rule

$$D(fs) = df \otimes s + fDs$$

for all $s \in \Gamma(M, \mathscr{C}^\infty(E))$ and all $f \in \mathscr{C}^\infty(M)$.

1.5.2 Example. Let $M \times \mathbb{C}^r \to M$ be the trivial vector bundle. The map

$$d : \Gamma(M, \mathscr{C}^\infty(M \times \mathbb{C}^r)) \to \Gamma(M, \mathscr{C}^\infty(\Lambda_M^1 \otimes M \times \mathbb{C}^r)$$

defined by

$$d(s^1, ..., s^r) := (ds^1, ..., ds^r)$$

is a connection, called the *trivial connection.* ◇

1.5.3 Proposition. *If D_1 and D_2 are two connections for a vector bundle $E \to M$ then there exists*

$$A \in \Gamma(M, \mathscr{C}^\infty(T_M^* \otimes \mathrm{Hom}(E, E)))$$

such that $(D_1 - D_2)s = As$.

Proof. By the Leibniz rule the linear map $\mathscr{A} := D_1 - D_2$ is a map of $\mathscr{C}^\infty$-modules, i.e., it satisfies $\mathscr{A}(fs) = f\mathscr{A}s$ for all sections $s \in \Gamma(M, \mathscr{C}^\infty(E))$ and all functions $f \in \mathscr{C}^\infty(M)$. Choosing a local frame $v_1, ..., v_r$ for E and a dual frame $\alpha^1, ..., \alpha^r$ for E^*, i.e., $\left\langle f_i, \alpha^j \right\rangle = \delta_i^j$, where the pairing is the natural one between E and E^*, we define the 1-forms A_i^j by the formula

$$\mathscr{A} v_i = A_i^j \otimes v_j \text{ (with summation convention in force, as usual).}$$

Then, with $A := A_i^j \otimes v_j \otimes \alpha^i$, for any section s, written locally as $s = f^i v_i$, we have

$$\mathscr{A}(s) = f^i \mathscr{A} v_i = f^i A_i^j \otimes v_j = f^i \langle A, v_i \rangle = \langle A, s \rangle,$$

as claimed. □

1.5.4 Remark. Proposition 1.5.3 says that the space of connections on a vector bundle is an affine space modeled on the vector space $\Gamma(M, \mathscr{C}^\infty(T_M^* \otimes \mathrm{Hom}(E, E)))$. ◇

Connection matrix

As mentioned in Example 1.2.2, a frame $e_1, ..., e_r$ of $E \to M$ over $U \subset M$ determines an isomorphism of $E|_U$ with the trivial bundle. By Example 1.5.2 $E|_U \to U$ has a canonical connection induced by the trivial connection of Example 1.5.2. The isomorphism induced by the frame shows that this connection is given by $d(s^i e_i) := ds^i \otimes e_i$. (This connection depends on the frame, though it is traditional to use notation that ignores this dependence.)

If D be any other connection for $E \to M$ then by Proposition 1.5.3

$$D|_U - d$$

defines a unique element $\theta \in \Gamma(U, \mathscr{C}^\infty(E|_U))$ such that $Ds = ds + \langle \theta, s \rangle$. In terms of the frame $e_1, ..., e_r$ for $E \to M$, θ is the $r \times r$ matrix of 1-forms θ_i^j defined by

$$De_i = \theta_i^j \otimes e_j.$$

The matrix θ is called the *connection matrix of D associated to the frame $e_1, ..., e_r$*.

Remark. It is useful to know how the connection matrix transforms when we use a different frame. Two frames $\tilde{e}_1, ..., \tilde{e}_r$ over $\tilde{U}$ and $e_1, ..., e_r$ over U are related by a map $G = (g_i^j) : U \cap \tilde{U} \to GL(n)$:

$$\tilde{e}_i = g_i^j e_j, \quad 1 \le i \le n.$$

We then have

$$\begin{aligned} D(s^i e_i) &= ds^i e_i + s^i \theta_i^j e_j \\ &= d(\tilde{s}^j g_j^i) e_i + \tilde{s}^j g_j^k \theta_k^\ell (g^{-1})_\ell^i \tilde{e}_i \\ &= d(\tilde{s}^j)\tilde{e}_j + \tilde{s}^j \left(dg_j^k (g^{-1})_k^i + \tilde{s}^j g_j^k \theta_k^\ell (g^{-1})_\ell^i \right) \tilde{e}_i. \end{aligned}$$

Thus the transformation rule is

$$\tilde{\theta} = (dG)G^{-1} + G\theta G^{-1}.$$

◇

1.5.2 Induced connections

Connection for the dual bundle

A connection D for a vector bundle $E \to M$, induces the connection $D^\vee$ for the dual vector bundle $E^* \to M$ as follows. For local sections s for E and α for E^* the pairing $\langle s, \alpha \rangle$ is a function on M. The dual connection $D^\vee$ is defined by requiring that

$$d \langle s, \alpha \rangle = \langle Ds, \alpha \rangle + \left\langle s, D^\vee \alpha \right\rangle. \tag{1.8}$$

For every local section s we have

$$\left\langle s, D^\vee \alpha \right\rangle = d \langle s, \alpha \rangle - \langle Ds, \alpha \rangle,$$

and the right hand side is linear in α. Moreover,

$$\begin{aligned}\left\langle s, D^\vee(f\alpha)\right\rangle &= d\langle s, f\alpha\rangle - \langle Ds, f\alpha\rangle = d\langle fs, \alpha\rangle - \langle fDs, \alpha\rangle \\ &= d\langle fs, \alpha\rangle + \langle df\otimes s, \alpha\rangle - \langle D(fs), \alpha\rangle \\ &= \langle df\otimes s, \alpha\rangle - \left\langle fs, D^\vee\alpha\right\rangle = \langle s, df\otimes\alpha\rangle - \left\langle s, fD^\vee\alpha\right\rangle.\end{aligned}$$

Therefore $D^\vee$ is indeed a connection. And if we fix a frame $e_1, ..., e_r$ for E and denote by $\alpha^1, ..., \alpha^r$ its dual frame, then we have

$$0 = d\delta_i{}^j = d\left\langle e_i, \alpha^j\right\rangle = \left\langle \theta(D)_i{}^k e_k, \alpha^j\right\rangle + \left\langle e_i, \theta(D^\vee)_\ell{}^j\alpha^\ell\right\rangle = \theta(D)_i{}^j + \theta(D^\vee)_i{}^j.$$

Thus we see that the connection $D^\vee$ is completely determined by (1.8).

Product connections

Let E_1, E_2 be two vector bundles over a manifold M. If D_1 and D_2 are connections for E_1 and E_2 respectively, there is a natural definition of connection D for any bilinear product of E_1 and E_2. Such a definition is determined by requiring the formula

$$D(s_1\times s_2) = (D_1s_1)\times s_2 + s_1\times(D_2s_2)$$

where $\times$ denotes the product in question, e.g., tensor, symmetric or wedge products. One can then inductively define the product of finitely many connections.

1.5.5 Example (Induced connection on $\mathrm{Hom}(E, E)$). Given a vector bundle E with connection D_E, one can consider the vector bundle

$$\mathrm{Hom}(E, E) = E\otimes E^*.$$

The induced connection $D_{\mathrm{Hom}(E,E)}$ is the product connection; it is given on the indecomposable elements $v\otimes\xi\in E\otimes E^*$ as

$$D_{\mathrm{Hom}(E,E)}(v\otimes\xi) = (D_Ev)\otimes\xi + v\otimes(D_E^\vee\xi).$$

If we fix a frame $v_1, ..., v_r$ for E and a dual frame $\xi^1, ..., \xi^r$ for E^* then by (1.8)

$$D_{\mathrm{Hom}(E,E)}v_i\otimes\xi^j = \theta(D_E)_i^k v_k\otimes\xi^j + \theta(D_E^\vee)_\ell^j v_i\otimes\xi^\ell = \theta(D_E)_i^k v_k\otimes\xi^j - \theta(D_E)_\ell^j v_i\otimes\xi^\ell.$$

There is some cancellation in the sums above, but not complete cancellation unless the rank of E is 1. For instance, in the rank 2 case we have

$$D_{\mathrm{Hom}(E,E)}v_1\otimes\xi^1 = \theta(D_E)_1^2 v_2\otimes\xi^1 - \theta(D_E)_2^1 v_1\otimes\xi^2,$$

so the terms involving $v_1\otimes\xi^1$ have cancelled out. ◇

1.5.6 Example (Induced connections for determinant bundles). Let $E \to M$ be a vector bundle of rank r and let a connection D_E for E be given. Consider the complex line bundle

$$\det E \to M,$$

the top exterior power of E. Equivalently the transition functions for $\det E$ are just the determinants of the corresponding transition functions for E. Fix a frame $e_1, ..., e_r$ for E. The wedge product $e_1 \wedge ... \wedge e_r$ is a frame for $\det E$. By the skew symmetry of the wedge product

$$e_1 \wedge ... \wedge De_j \wedge ... \wedge e_r = e_1 \wedge ... \wedge \theta_j^k e_k \wedge ... \wedge e_r = \theta_j^k \delta_{jk} e_1 \wedge ... \wedge e_j \wedge ... \wedge e_r,$$

and thus

$$D_{\det E}(e_1 \wedge ... \wedge e_r) = \theta_j^j e_1 \wedge ... \wedge e_r,$$

i.e., the connection matrix for $D_{\det E}$ is the trace of the connection matrix for D_E. $\diamond$

1.5.3 Connections with additional symmetry

Metric Compatibility

Let $E \to M$ be endowed with a metric, say g.

1.5.7 Definition. We say that a connection D is compatible with the metric g if

$$d(g(s,t)) = g(Ds,t) + g(s,Dt)$$

for all local sections s, t of $E \to M$. $\diamond$

There are many metric-compatible connections for a Hermitian vector bundle. If D_1 and D_2 are metric-compatible connections for $E \to M$ then $\Theta := D_1 - D_2 \in \Gamma(M, T_M^* \otimes \mathrm{Hom}(E,E))$ satisfies

$$g(\Theta s, t) + g(s, \Theta t) = 0,$$

i.e., Θ is anti-symmetric (or anti-Hermitian if g is a Hermitian metric) with respect to g.

1.5.8 Proposition. *Let $E \to M$ be a smooth vector bundle with metric g. The space of metric-compatible connections is an affine space modeled on the space $\Gamma(M, \mathscr{C}^\infty(M \otimes \mathcal{A}_g(E)))$ of smooth sections of the vector bundle whose fibers consist of g-anti-symmetric (resp. anti-Hermitian) endomorphisms of E.*

Proof. As we have already demonstrated, any two g-compatible connections must differ by a section of $\Gamma(M, \mathscr{C}^\infty(M \otimes \mathcal{A}_g(E)))$. To complete the proof we must show there is at least one metric-compatible connection.

First consider the local problem: Choose an open set U for where there is a frame $\{e_1, ..., e_r\}$ for $E|_U$ and write $s = s^i e_i$, $t = t^i e_i$ and $g_{ij} = g(e_i, e_j)$. Define $D_U s :=$

$\left(ds^i + s^j\left(\frac{1}{2}g^{ik}dg_{kj}\right)\right) \otimes e_i$, where g^{ij} are the entries of the inverse matrix of g (which is the matrix of a dual metric in the dual frame). Then

$$\begin{aligned} g(D_U s, t) + g(s, D_U t) &= \left(ds^i + s^j\left(\frac{1}{2}g^{ik}dg_{kj}\right)\right) t^\ell g_{i\ell} + s^i\left(dt^\ell + t^j\left(\frac{1}{2}g^{\ell k}dg_{kj}\right)\right) g_{i\ell} \\ &= ds^i t^\ell g_{i\ell} + s^i dt^\ell g_{i\ell} + s^i t^\ell dg_{i\ell} = d(g(s,t)). \end{aligned}$$

Now take an open cover $\{U_j\}_{j\in J}$ of M and a partition of unity $\{\chi_\alpha\}_{\alpha\in A}$ subordinate to this cover. Choose a map $\mu : A \to J$ such that χ_α is supported in $U_{\mu(\alpha)}$ for all $\alpha \in A$. Then

$$D := \sum_{\alpha\in A} \chi_\alpha D_{U_{\mu(\alpha)}} \tag{1.9}$$

is g-compatible, and the proof is complete. □

Remark. If we view our metric g as a section of $E^* \otimes E^* \to M$, then a connection on E induces a connection of $E^* \otimes E^*$ via a combination of the product and dual constructions discussed in the previous paragraph. For a metric and connection D,

$$d(g(s,t)) = Dg(s,t) + g(Ds,t) + g(s,Dt).$$

Thus metric compatibility means that $Dg = 0$ (with respect to the induced connection). The discussion just preceeding the remark then tells us that the formula for the induced connection with respect to some frame chosen for E is

$$(Dg)_{ij} = dg_{ij} - \theta_i^{\ k} g_{kj} - g_{i\ell}\theta_j^{\ \ell}.$$

We will meet this induced connection again in Section 1.5.8. ◇

Remark (Induced metrics and induced metric connections). If $E_1, E_2 \to M$ are vector bundles with metrics g_1, g_2 respectively then the *induced metric* $g_1 \oplus g_2$ on $E_1 \oplus E_2 \to M$ is taken to be the one defined by

$$g_1 \oplus g_2(v_1 \oplus v_2, w_1 \oplus w_2) := g_1(v_1, w_1) + g_2(v_2, w_2).$$

In other words, the subbundles $E_1 \oplus 0 \cong E_1$ and $0 \oplus E_2 \cong E_2$ carry the metrics g_1 and g_2 respectively, and are orthogonal with respect to $g_1 \oplus g_2$. If connections D_1, D_2 with $D_i g_i = 0$ are given, the product connection $D = D_1 \oplus D_2$ is compatible with $g = g_1 \oplus g_2$. (In general, there will be other connections that are compatible with g.)

Similarly, for the tensor product $E_1 \otimes E_2$ one has the induced metric $g_1 \otimes g_2$ defined by

$$g_1 \otimes g_2(v_1 \otimes v_2, w_1 \otimes w_2) := g_1(v_1, w_1) \cdot g_2(v_2, w_2)$$

on indecomposible tensors, and extended by linearlity. The product connection $D_1 \otimes D_2$ is then compatible with the metric $g_1 \otimes g_2$. ◇

Symmetric connections

On any smooth manifold M there is a splitting

$$T_M^* \otimes T_M^* = \mathrm{Sym}^2(T_M^*) \oplus \Lambda^2(T_M^*),$$

Accordingly, every connection $D : \Gamma(M, \mathscr{C}^\infty(T_M^*)) \to \Gamma(M, \mathscr{C}^\infty(T_M^* \otimes T_M^*))$ for the cotangent bundle splits as

$$D = D^S + D^\Lambda.$$

On any manifold the exterior derivative d sends 1-forms to 2-forms and satisfies the Leibniz rule with respect to the wedge-product. We can thus identify a class of connections for the cotangent bundle (and hence for all connections induced on any vector bundle obtained from T_M^* via multilinear operations, e.g., T_M and $(T_M^*)^{\otimes r} \otimes (T_M)^{\otimes s}$) by requiring that $D^\Lambda = d$.

1.5.9 Definition. A connection D for T_M^* is said to be *symmetric* if $D^\Lambda = d$. ⋄

We have the following analogue of Proposition 1.5.8.

1.5.10 Proposition. *Let M be a smooth manifold. The space of symmetric connections is an affine space modeled on the space $\Gamma(M, \mathscr{C}^\infty(\mathrm{Sym}^2(T_M^*)))$ of symmetric covariant 2-tensors.*

Proof. The difference of two connections is a section of the endomorphism bundle $\mathscr{C}^\infty(T_M^* \otimes T_M^*)$, and from the definition of symmetric connection it is clear that such a difference is symmetric. We must show that this affine space is non-empty. As in the proof of Proposition 1.5.8 we can use partitions of unity to reduce the problem to the local existence of affine connections. The local existence of a symmetric connection is trivial (pun intended): in a neighborhood in which T_M^* is trivialized by a local frame $\alpha^1, ..., \alpha^m$ satisfying that $d\alpha^i = 0$ for $1 \le i \le m$, set $D(f_i\alpha^i) := df_i \otimes \alpha^i$. Then

$$D^\Lambda(f_i\alpha^i) := df_i \wedge \alpha^i = d(f_i\alpha^i),$$

as required. □

In every coordinate system $x = (x^1, ..., x^m)$ a connection D for T_M^* determines a collection of functions C^i_{jk}, $1 \le i, j, k \le m$ by the relation

$$D(dx^i) = C^i_{jk} dx^k \otimes dx^j.$$

The functions C^i_{jk} are classically called the *Christoffel symbols* of D in the coordinate system $x = (x^1, ..., x^m)$. The skew-symmetric part of the tensor $D(f_i dx^i)$ is

$$\begin{aligned}
\Lambda^2\left(D(f_i dx^i)\right) &= \Lambda^2\left(\left(\frac{\partial f_j}{\partial x^k} + f_i C^i_{jk}\right) dx^k \otimes dx^j\right) \\
&= \frac{1}{2}\left(\frac{\partial f_j}{\partial x^k} - \frac{\partial f_k}{\partial x^j}\right) dx^k \otimes dx^j + f_i \frac{\left(C^i_{jk} - C^i_{kj}\right)}{2} dx^k \otimes dx^j \\
&= d(f_i dx^i) + f_i C^i_{jk} dx^k \wedge dx^j.
\end{aligned}$$

It follows that D is symmetric if and only if its Christoffel symbols satisfy

$$C^i_{jk} = C^i_{kj}.$$

It is often useful to formulate the notion of symmetric connection for the tangent bundle. (This is the most commonly encountered approach in Riemannian geometry.) We say that a connection for T_M *symmetric* if it is the dual of a symmetric connection for T^*_M. If ∇ is a connection for T_M dual to a given connection D for T^*_M and we write

$$\nabla \frac{\partial}{\partial x^j} = \Gamma^i_j \frac{\partial}{\partial x^i} = \Gamma^i_{jk} dx^k \otimes \frac{\partial}{\partial x^i}$$

then the functions $\Gamma^i_{jk} = \left\langle \nabla_{\frac{\partial}{\partial x^k}}(\frac{\partial}{\partial x^j}), dx^i \right\rangle$ (called the Christoffel symbols for ∇) satisfy

$$\Gamma^i_{jk} = \left\langle \nabla_{\frac{\partial}{\partial x^k}}(\tfrac{\partial}{\partial x^j}), dx^i \right\rangle = -\left\langle \tfrac{\partial}{\partial x^j}, D_{\frac{\partial}{\partial x^k}}(dx^i) \right\rangle = -C^i_{jk}.$$

Thus ∇ is symmetric if and only if its connection matrix Γ^i_{jk} is symmetric, i.e.,

$$\Gamma^i_{jk} = \Gamma^i_{kj}.$$

Finally observe that if $\xi = \xi^i \frac{\partial}{\partial x^i}$ and $\eta = \eta^i \frac{\partial}{\partial x^i}$ are local vector fields, then

$$\nabla_\xi \eta - \nabla_\eta \xi = \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} + \xi^i \eta^j (\Gamma^k_{ij} - \Gamma^k_{ji}) \frac{\partial}{\partial x^k},$$

and thus ∇ is symmetric if and only if $\nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta]$.

The map $T_\nabla : \Gamma(M, \mathscr{C}^\infty(T_M \otimes T_M)) \to \Gamma(M, \mathscr{C}^\infty(T_M))$ defined by

$$T_\nabla(\xi \otimes \eta) := \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]$$

is called the *torsion* of the connection ∇. In this terminology a connection is by definition symmetric if its torsion vanishes. It is easy to see that T_∇ is $\mathscr{C}^\infty(M)$-linear, and therefore determines a unique section of $\Gamma(M, \mathscr{C}^\infty(T_M \otimes T^*_M \otimes T^*_M))$, abusively denoted T_∇.

Levi-Civita connections

On a Riemannian manifold there is exactly one symmetric, metric-compatible connection.

1.5.11 Theorem (Levi-Civita). *On a Riemannian manifold (M, γ) there exists a unique symmetric connection compatible with γ. In terms of the dual metric $g := \gamma^\vee$ for T^*_M, there exists a unique connection D such that*

$$d(g(s, s')) = g(Ds, s') + g(s, Ds') \quad \textit{and} \quad D^\Lambda = d.$$

Proof. The most natural coordinate-free proof is obtained by looking for the dual connection ∇. Let ξ_1, ξ_2, ξ_3 be vector fields. For any connection ∇ we have

$$\xi_i g(\xi_j, \xi_k) = g(\nabla_{\xi_i}\xi_j, \xi_k) + g(\xi_j, \nabla_{\xi_i}\xi_k) + (\nabla_{\xi_i} g)(\xi_j, \xi_k),$$

where in the last term on the right the connection is the one induced on metrics by ∇, which we abusively denote with the same letter. Thus

$$\begin{aligned}&\xi_1 g(\xi_2,\xi_3) - \xi_3 g(\xi_1,\xi_2) + \xi_2 g(\xi_3,\xi_1)\\ &= g(\nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1, \xi_3) - g(\nabla_{\xi_3}\xi_1 - \nabla_{\xi_1}\xi_3, \xi_2) + g(\nabla_{\xi_2}\xi_3 - \nabla_{\xi_3}\xi_2, \xi_1) + 2g(\nabla_{\xi_2}\xi_1, \xi_3)\\ &\qquad +(\nabla_{\xi_1} g)(\xi_2,\xi_3) - (\nabla_{\xi_3} g)(\xi_1,\xi_2) + (\nabla_{\xi_2} g)(\xi_3,\xi_1),\end{aligned}$$

i.e.,

$$\begin{aligned}&2g(\nabla_{\xi_2}\xi_1, \xi_3)\\ &= \xi_1 g(\xi_2,\xi_3) - \xi_3 g(\xi_1,\xi_2) + \xi_2 g(\xi_3,\xi_1) - (g([\xi_1,\xi_2],\xi_3) - g([\xi_3,\xi_1],\xi_2) + g([\xi_2,\xi_3],\xi_1))\\ &\qquad +(\nabla_{\xi_1} g)(\xi_2,\xi_3) - (\nabla_{\xi_3} g)(\xi_1,\xi_2) + (\nabla_{\xi_2} g)(\xi_3,\xi_1)\\ &\qquad\quad + g(\langle T_\nabla, \xi_1\otimes\xi_2\rangle, \xi_3) - g(\langle T_\nabla, \xi_3\otimes\xi_1\rangle, \xi_2) + g(\langle T_\nabla, \xi_2\otimes\xi_3\rangle, \xi_1),\end{aligned}$$

where T_∇ is the torsion tensor. If the connection ∇ is metric compatible and symmetric then the last two lines vanishe, and hence any symmetric connection compatible with g must satisfy the formula

$$\begin{aligned}g(\nabla_{\xi_2}\xi_1, \xi_3) &= \tfrac{1}{2}\left(\xi_1 g(\xi_2,\xi_3) - \xi_3 g(\xi_1,\xi_2) + \xi_2 g(\xi_3,\xi_1)\right)\\ &\qquad -\tfrac{1}{2}\left(g([\xi_1,\xi_2],\xi_3) - g([\xi_3,\xi_1],\xi_2) + g([\xi_2,\xi_3],\xi_1)\right).\end{aligned} \tag{1.10}$$

Since the right hand side of (1.10) is independent of the connection, there can be at most one symmetric connection compatible with g. On the other hand, the formula (1.10) determines such a connection, so there is at least one symmetric connection compatible with g. □

Complex connection

On a complex manifold X we have a splitting

$$T_X^* \otimes \mathbb{C} = T_X^{*1,0} \oplus T_X^{*0,1}.$$

It follows that for a complex vector bundle $E \to X$, a connection D splits as

$$D = D^{1,0} + D^{0,1}. \tag{1.11}$$

If the vector bundle $E \to X$ is moreover *holomorphic* then there is a $\bar\partial$-operator (see Page 24)

$$\bar\partial : \Gamma(X, \mathscr{C}^\infty(E)) \to \Gamma(X, \mathscr{C}^\infty(\Lambda_X^{0,1} \otimes E)).$$

We can use $\bar\partial$ to define a class of connections for a holomorphic vector bundle as follows.

1.5.12 Definition. A connection D for a holomorphic vector bundle $E \to X$ is said to be *complex* if $D^{0,1} = \bar\partial$ in terms of the splitting (1.11).

The fundamental theorem of holomorphic Hermitian geometry

The basic result about connections for holomorphic Hermitian vector bundles is the following analogue of Levi-Civita's theorem.

1.5.13 Theorem (Chern). *On a holomorphic Hermitian vector bundle there exists a unique complex connection compatible with the Hermitian metric.*

Proof. Suppose D is a unique complex connection compatible with the Hermitian metric of a holomorphic Hermitian vector bundle (E, h). Metric compatibility means that $d(h(s, \sigma)) = h(Ds, \sigma) + h(s, D\sigma)$. Taking the $(1, 0)$-part gives

$$\partial(h(s, \sigma)) = h(D^{1,0}s, \sigma) + h(s, D^{0,1}\sigma).$$

Since $D^{0,1} = \bar{\partial}$ we have

$$h(D^{1,0}s, \sigma) = \partial(h(s, \sigma)) - h(s, \bar{\partial}\sigma).$$

The right hand side is canonically defined on a holomorphic Hermitian vector bundle, i.e., it does not depend on any connection. We can therefore use this identity to *define* $D^{1,0}$. Thus both existence and uniqueness are proved at once. □

1.5.14 Definition. The unique metric-compatible complex connection for a Hermitian holomorphic vector bundle (E, h) is called the *Chern connection* for (E, h).

1.5.15 Remark. In a frame $e_1, ..., e_r$, if we write $s = s^\alpha e_\alpha$, $\sigma = \sigma^\beta e_\beta$ and $h_{\alpha\bar{\beta}} := h(e_\alpha, \bar{e}_\beta)$ then

$$\begin{aligned}\partial(h(s, \sigma)) - f(s, \bar{\partial}\sigma) &= \partial\left(s^\alpha \bar{\sigma}_\beta h_{\alpha\bar{\beta}}\right) - s^\alpha \overline{(\bar{\partial}\sigma_\beta)} h_{\alpha\bar{\beta}} \\ &= \left((\partial s^\alpha)\, h_{\alpha\bar{\beta}} + s^\alpha \partial h_{\alpha\bar{\beta}}\right) \bar{\sigma}_\beta \\ &= \left((\partial s^\alpha) + s^\gamma \partial h_{\gamma\bar{\beta}} h^{\alpha\bar{\delta}}\right) \bar{\sigma}_\beta h_{\alpha\bar{\beta}}.\end{aligned}$$

Therefore D is given by the formula

$$D^{1,0}s = \left(\partial s^\alpha + s^\gamma h^{\alpha\bar{\delta}} \partial h_{\gamma\bar{\delta}}\right) e_\alpha \quad \text{and} \quad Ds = \left(\partial s^\alpha + s^\gamma h^{\alpha\bar{\delta}} \partial h_{\gamma\bar{\delta}} + \bar{\partial} s^\alpha\right) e_\alpha.$$

One often finds the abusive notation $D^{1,0} = \partial + (\partial h)h^{-1}$. ◇

1.5.4 The Kähler condition

The definition of Kähler metric

The tangent bundle T_X of a complex manifold X with Hermitian Riemannian metric g carries two natural connections. One connection is the Chern connection for the holomorphic Hermitian vector bundle (T_X, g), and the other connection is the Levi-Civita connection for (T_X, g). (Here g is viewed both as a Hermitian metric on $T_X^{1,0}$ and as a Riemannian metric on T_X, via the correspondence $s^{1,0} : T_X \to T_X^{1,0}$ discussed in Section 1.3.) In general, these two connections are different.

1.5.16 DEFINITION. A metric g for which the Chern and Levi-Civita connections agree is said to be Kähler. A complex manifold admitting a Kähler metric is called a Kähler manifold.

There are several equivalent ways to formulate the definition of a Kähler metric.

TORSION FORMULATION

Let $D = D^{1,0} + \bar\partial$ be the Chern connection for $(T^{*1,0}_X, g)$. Letting $\Lambda(\beta)$ denote the skew symmetric part of a tensor β, we obtain a well-defined operator

$$\mathscr{T}(g) := \Lambda \circ (\partial - D^{1,0}) = \partial - \Lambda \circ D^{1,0} \in \mathrm{Hom}(T^{*1,0}_X, \Lambda^{2,0}_X),$$

called the *torsion* of g. Note that since $D^{0,1} = \bar\partial$,

$$\mathscr{T}(g) = \Lambda \circ (d - D).$$

We therefore immediately obtain the following proposition.

1.5.17 PROPOSITION. *The metric g is Kähler if and only if $\mathscr{T}(g) = 0$.*

It is worth having a look at the local form of the torsion. Let z be a local coordinate system. If f is a 1-form on X then the Chern connection D is given by the formula

$$Df = D(f_i dz^i) = df_i \otimes dz^i - f_j g_{i\bar\ell} \frac{\partial g^{j\bar\ell}}{\partial z^k} dz^k \otimes dz^i.$$

Therefore

$$\Lambda(df - Df) = \Lambda\left(f_j g_{i\bar\ell} \frac{\partial g^{j\bar\ell}}{\partial z^k} dz^k \otimes dz^i\right).$$

Since $g_{i\bar\ell} g^{j\bar\ell} = \delta^j_i$, $\frac{\partial}{\partial z^k}\left(g_{i\bar\ell} g^{j\bar\ell}\right) = \frac{\partial g_{i\bar\ell}}{\partial z^k} g^{j\bar\ell} + g_{i\bar\ell}\frac{\partial g^{j\bar\ell}}{\partial z^k}$ and hence

$$\Lambda\left(f_j g_{i\bar\ell} \frac{\partial g^{j\bar\ell}}{\partial z^k} dz^k \otimes dz^i\right) = -\Lambda\left(f_j g^{j\bar\ell} \frac{\partial g_{i\bar\ell}}{\partial z^k} dz^k \otimes dz^i\right) = f^{\sharp_g} \lrcorner \partial\omega_{g^*},$$

where where g^* denotes the metric for $T^{1,0}_X$ dual to g, $f^{\sharp_g}$ is the $(0,1)$-vector field on X obtained from the $(1,0)$-form f via the identity $f(v) = \omega_{g^*}(v, f^{\sharp_g})$, and $\lrcorner$ denotes contraction. Therefore the torsion operator is given by

$$\mathscr{T}(g)f := \left(f^{\sharp_g}\right) \lrcorner \partial\omega_{g^*}. \tag{1.12}$$

Let $E \to X$ be a smooth (and not necessarily holomorphic) vector bundle. Closely related to the torsion $\mathscr{T}(g)$ is the following action on E-valued $(0,q)$-forms.

1.5.18 DEFINITION. We define the pointwise operator

$$\tau_g := -\left(\Lambda_g \partial\omega_g\right)^{\sharp_g} : \Lambda^{0,q}_X \otimes E \to \Lambda^{p,q-1}_X \otimes E.$$

Locally with respect to a holomorphic coordinate system z and a local frame $a_1, ..., e_r$ for E, the action is given by

$$\tau_g \varphi := -g^{i\bar j} g^{k\bar\ell} \frac{\partial g_{i\bar\ell}}{\partial z^k} \varphi^\alpha_{\bar j \bar L} d\bar z^L \otimes e_\alpha,$$

where $L = (\ell_1, ..., \ell_{q-1})$.

Symplectic formulation

1.5.19 Proposition. *A Hermitian Riemannian metric γ is Kähler if and only if the associated metric $(1,1)$-form $\omega = \omega_\gamma$ is closed.*

Proof. In view of Proposition 1.5.17 and Formula (1.12), γ is Kähler if and only if $\partial\omega_\gamma = 0$. Since ω_γ is a real $(1,1)$-form, $\partial\omega_\gamma = 0$ if and only if $d\omega_\gamma = 0$. □

1.5.20 Remark. Since every 2-form on a real surface is closed, we see that every Hermitian metric on a Riemann surface is Kähler. ◇

Locally Euclidean formulation

A very useful formulation of the Kähler condition is contained in the following theorem.

1.5.21 Theorem. *The metric g is Kähler if and only if every point of X lies in a coordinate chart with coordinates z so that*

$$g = \sum_\alpha dz^\alpha \cdot \ d\bar{z}^\alpha + O(|z|^2).$$

The coordinates referred to in the theorem are called *Kähler coordinates.*

Proof. If two $(1,1)$-forms ω_1 and ω_2 defined in a neighborhood of 0 in $\mathbb{C}^n$ have the same Taylor expansion up to second order then

$$(d\omega_1)_0 = (d\omega_2)_0.$$

In particular, if g is locally Euclidean to second order in some holomorphic coordinate system then $d\omega_g = 0$, and thus g is Kähler by Proposition 1.5.19.

Conversely, suppose ω is the $(1,1)$-form associated to a Kähler metric g, and let z be local coordinates such that

$$g_{i\bar{j}}(0) = \delta_{i\bar{j}} \tag{1.13}$$

(It is easy to find such coordinates.) Then the Taylor expansion of ω with respect to z is

$$\omega = \sqrt{\text{-}1}\left(\delta_{i\bar{j}} + a_{i\bar{j}k}z^k + a_{i\bar{j}\bar{\ell}}\bar{z}^\ell + O(|z|^2)\right) dz^i \wedge d\bar{z}^j.$$

The two properties of the Taylor coefficients $a_{i\bar{j}k}, a_{i\bar{j}\bar{k}}$ are

(a) $g_{i\bar{j}} = \overline{g_{j\bar{i}}} \Rightarrow a_{i\bar{j}\bar{k}} = \overline{a_{j\bar{i}k}}$,

(b) $d\omega = 0 \Rightarrow a_{i\bar{j}k} = a_{k\bar{j}i}$.

Since the result we seek is regarding a second order Taylor expansion, it suffices to find a quadratic biholomorphic local coordinate transformation. That is to say, we seek coordinates w defined by

$$z^k = w^k + \frac{1}{2}b^k_{\ell m}w^\ell w^m$$

(which evidently do not modify condition (1.13)) such that

$$\omega = \frac{\sqrt{-1}}{2}\left(\delta_{i\bar{j}} + O(|w|^2)\right) dw^i \wedge d\bar{w}^j.$$

Since $w^\ell w^m = w^m w^\ell$, we may assume that $b^i_{jk} = b^i_{kj}$. Then

$$dz^k = dw^k + b^k_{\ell m}w^\ell dw^m,$$

and we have

$$\begin{aligned}
-\sqrt{-1}\omega &= \delta_{i\bar{j}}\left(dw^i + b^i_{\ell m}w^\ell dw^m\right) \wedge \left(d\bar{w}^j + \overline{b^j_{rs}}\bar{w}^r d\bar{w}^s\right) \\
&\quad + \left(a_{i\bar{j}k}w^k + a_{i\bar{j}\bar{k}}\bar{w}^k\right) dw^i \wedge d\bar{w}^j + O(|w|^2) \\
&= \left(\delta_{i\bar{j}} + (a_{i\bar{j}k} + \delta_{\ell\bar{j}}b^\ell_{ki})w^k + (a_{i\bar{j}\bar{k}} + \overline{\delta_{m\bar{i}}b^m_{kj}})\bar{w}^k + O(|w|^2)\right) dw^i \wedge d\bar{w}^j.
\end{aligned}$$

Thus, if we set $b^j_{ki} = -\delta^{j\bar{m}}a_{i\bar{m}k}$ (which is symmetric in ki by (a)) then

$$\overline{\delta_{mi}b^m_{kj}} = -\overline{\delta_{mi}\delta^{m\bar{n}}a_{j\bar{n}k}} = -\delta^{n\bar{i}}\overline{a_{j\bar{n}k}} = -a_{i\bar{j}\bar{k}}.$$

Thus

$$\omega = \sqrt{-1}\delta_{i\bar{j}}dw^i \wedge d\bar{w}^j + O(|w|^2),$$

so g is Euclidean to second order. □

1.5.5 Twisted Exterior Derivatives of twisted forms

Symmetric connections and exterior derivatives

Consider a differential 1-form $\alpha = \alpha_i dx^i$ on a manifold M. For a connection ∇ for $T^*_M \to M$,

$$\nabla\alpha = \frac{\partial\alpha_i}{\partial x^j}dx^j \otimes dx^i + \alpha_k\theta^k_{ij}dx^j \otimes dx^i,$$

where θ is the connection matrix of ∇. As we discussed earlier,

$$\Lambda^2(\nabla\alpha) = \frac{\partial\alpha_i}{\partial x^j}dx^j \wedge dx^i + \alpha_k\theta^k_{ij}dx^j \wedge dx^i. \tag{1.14}$$

We then introduced the notion of symmetric connection; by definition, a connection ∇ is symmetric if and only if

$$d\alpha = \Lambda^2(\nabla\alpha)$$

for any 1-form α. Of course, the formula (1.14) shows that ∇ is symmetric if and only if $\theta_{ij}^k = \theta_{ji}^k$.

Now suppose β is a differential r-form, i.e., a section of the product bundle $\Lambda^r(T_M^*)$. For a connection ∇ for $T_M^* \to M$, the product connection ∇_r acts on $\beta = \beta_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}$ (with our convention that β_I is skew-symmetric in I) by

$$\nabla_r \beta = \left(\frac{\partial \beta_I}{\partial x^{i_o}} + \sum_{j=1}^{r} \beta_{i_1 \cdots (\ell)_j \cdots i_r} \theta_{i_o i_j}^{\ell} \right) dx^{i_o} \otimes dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

where the notation $(\ell)_j$ means that ℓ replaces i_j. If we take the $(r+1)^{\text{th}}$ skew-symmetric part of $\nabla_r \beta$ (thought of as an $(r+1)$-tensor) we obtain

$$\Lambda^{r+1}(\nabla_r \beta) = \left(\frac{\partial \beta_I}{\partial x^{i_o}} + \sum_{j=1}^{r} \beta_{i_1 \cdots (\ell)_j \cdots i_r} \theta_{i_o i_j}^{\ell} \right) dx^{i_o} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$

Thus we see that a connection ∇ is symmetric if and only if

$$d = \Lambda^{r+1} \circ \nabla_r$$

holds for any integer r with $1 \leq r \leq n$.

Twisted exterior derivative

Let $E \to M$ be a vector bundle with connection D. We can define a twisted version of the exterior derivative for sections of

$$\Gamma(M, \mathscr{C}^\infty(T_M^* \otimes E))$$

or *E-valued 1-forms*. This twisted exterior derivative should produce a E-valued 2-form.

As in the previous paragraph, we fix a connection ∇ for T_M^*. For a E-valued 1-form α we compute that

$$(\nabla \otimes D)\alpha = \left(\frac{\partial \alpha_i^\nu}{\partial x^j} + \alpha_i^\mu \omega_{\mu j}^\nu + \alpha_k^\nu \theta_{ij}^k \right) dx^j \otimes dx^i \otimes \mathrm{e}_\nu$$

and

$$\Lambda^2((\nabla \otimes D)\alpha) = \left(\frac{\partial \alpha_i^\nu}{\partial x^j} + \alpha_i^\mu \omega_{\mu j}^\nu + \alpha_k^\nu \theta_{ij}^k \right) dx^j \wedge dx^i \otimes \mathrm{e}_\nu,$$

where ω and θ are the Christoffel symbols for D and ∇ respectively. Again if the connection ∇ for T_M^* is symmetric, then the anti-symmetric part

$$\Lambda^2((\nabla \otimes D)\alpha) = \left(\frac{\partial \alpha_i^\nu}{\partial x^j} + \alpha_i^\mu \omega_{\mu j}^\nu \right) dx^j \wedge dx^i \otimes \mathrm{e}_\nu$$

is independent of the connection ∇. Similarly, if β is a E-valued r-form then

$$\Lambda^{r+1}(\nabla_r \otimes D)\alpha)$$

is a E-valued $r+1$-form, which is again independent of ∇ as soon as ∇ is symmetric.

1.5.22 Definition. Let $E \to M$ be a vector bundle with connection D and let ∇ be a symmetric connection for T_M^*. The operator $D_1 : \Gamma(M, \mathscr{C}^\infty(T_M^* \otimes E)) \to \Gamma(M, \mathscr{C}^\infty(\Lambda^2(T_M^*) \otimes E))$ defined by

$$D_1\alpha := \Lambda^2((\nabla \otimes D)\alpha)$$

(which is independent of ∇) is called the twisted exterior derivative associated to D. More generally, let ∇_r denote the induced product connection for $\Lambda^r(T_M^*) \to M$. The operator

$$D_r := \Lambda^{r+1} \circ (\nabla_r \otimes D) : \Gamma(M, \mathscr{C}^\infty(\Lambda^r(T_M^*) \otimes E)) \to \Gamma(M, \mathscr{C}^\infty(\Lambda^{r+1}(T_M^*) \otimes E))$$

is called the twisted r^{th} exterior derivative (for E-valued r-forms) associated to D. ◇

If $e_1, ..., e_r$ is a frame for E and $x^1, ..., x^m$ is a local coordinate system on M, then for a section $\sigma \in \Gamma(M, \Lambda^r(T_M^*) \otimes E)$ given locally by $\sigma = \sigma_I^\mu dx^I \otimes e_\mu$, one has (with $D = D_r$)

$$\begin{aligned} D\sigma &= \frac{\partial(\sigma_I^\mu)}{\partial x^j} dx^j \wedge dx^I \otimes e_\mu + \omega_\nu^\mu \wedge \sigma_I^\nu dx^I \otimes e_\mu \\ &= \frac{\partial(\sigma_I^\mu)}{\partial x^j} dx^j \wedge dx^I \otimes e_\mu + (-1)^r \sigma_I^\nu dx^I \wedge \omega_\nu^\mu \otimes e_\mu. \end{aligned}$$

Informally, we write

$$D\sigma = d\sigma + (-1)^r \sigma \wedge \omega.$$

1.5.23 Proposition. *Let $E \to M$ be a vector bundle with connection D. Then*

$$D_{r+k}(\alpha \wedge s) = d\alpha \wedge s + (-1)^k \alpha \wedge D_r s$$

for all $s \in \Gamma(M, \Lambda^r(T_M^*) \otimes E)$ and all $\alpha \in \Gamma(M, \Lambda^k(T_M^*))$.

The proof is left to the reader.

Computing the second exterior derivative of a section

For any section s of E, $Ds \in \Gamma(M, \mathscr{C}^\infty(T_M^* \otimes E))$ is a special case of a 1-form with values in E. In local coordinates $dx^1, ..., dx^m$ and a frame $e_1, ..., e_k$ for $E \to M$, we compute that

$$Ds = \left(\frac{\partial s^\mu}{\partial x^i} + s^\nu \omega_{\nu i}^\mu \right) dx^i \otimes e_\mu,$$

where $\omega^\mu_\nu e_\mu := De_\nu$. Taking a symmetric connection ∇ for T^*_M and setting $\theta^i_{jk} := \left\langle \frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^j}} dx^i \right\rangle$, we have

$$(\nabla \otimes D)(Ds) = (\nabla \otimes D)\left(\left(\frac{\partial s^\mu}{\partial x^i} + s^\nu \omega^\mu_{\nu i}\right) dx^i \otimes e_\mu\right)$$

$$= \left(\frac{\partial}{\partial x^j}\left(\frac{\partial s^\mu}{\partial x^i} + s^\nu \omega^\mu_{\nu i}\right) + \left(\frac{\partial s^\lambda}{\partial x^i} + s^\nu \omega^\lambda_{\nu i}\right)\omega^\mu_{\lambda j} + \left(\frac{\partial s^\mu}{\partial x^k} + s^\nu \omega^\mu_{\nu k}\right)\theta^k_{ji}\right) dx^j \otimes dx^i \otimes e_\mu$$

$$= \left(\frac{\partial^2 s^\mu}{\partial x^j \partial x^i} + \left(\frac{\partial s^\nu}{\partial x^j}\omega^\mu_{\nu i} + \frac{\partial s^\lambda}{\partial x^i}\omega^\mu_{\lambda j}\right) + \left(\frac{\partial s^\mu}{\partial x^k} + s^\nu \omega^\mu_{\nu k}\right)\theta^k_{ji}\right.$$

$$\left. + s^\nu\left(\frac{\partial \omega^\mu_{\nu i}}{\partial x^j} + \omega^\lambda_{\nu i}\omega^\mu_{\lambda j}\right)\right) dx^j \otimes dx^i \otimes e_\mu.$$

It follows from the symmetry of ∇ and of the first two terms in the third line that

$$\Theta(s) := \Lambda^2((D\otimes\nabla)Ds) = s^\nu\left(\frac{\partial \omega^\mu_{\nu i}}{\partial x^j} + \omega^\lambda_{\nu i}\omega^\mu_{\lambda j}\right) dx^j \wedge dx^i \otimes e_\mu = s^\nu(d\omega^\mu_\nu - \omega^\lambda_\nu \wedge \omega^\mu_\lambda)\otimes e_\mu. \tag{1.15}$$

We see again that Θ is independent of the choice of symmetric connection ∇, but perhaps more striking is the fact that the formula (1.15) does not depend on the derivative of s.

1.5.6 Curvature

The name *curvature* is both suggestive and vague. And indeed, there are many ways in which curvature gets used in mathematics.

When we turn to L^2 methods, our main interest will be to see how curvature affects changing the order of differentiation for mixed (covariant) partial derivatives. We will take advantage of the situation when this change of the order of differentiation gives rise to a positive operator.

In this section we will develop the basic properties of curvature, and apply them to get some topological consequences. We shall leave aside the geometric applications for now, but these will be central in many parts of the book.

The definition of curvature

1.5.24 Definition. Let $E \to M$ be a vector bundle with connection D and, in terms of some frame, connection matrix ω. The curvatures of (E, D) are the operators

$$\Theta_k := D_{k+1}D_k : \Gamma(M, \mathscr{C}^\infty(\Lambda^k(T^*_M)\otimes E)) \to \Gamma(M, \mathscr{C}^\infty(\Lambda^{k+2}(T^*_M)\otimes E)),$$

where D_j is the twisted exterior derivative associated to the connection D as in Definition 1.5.22.

Observe that if s is a E-valued k-form and f is a function then by Proposition 1.5.23

$$\Theta_k(fs) = D(fDs + df \wedge s) = fD \circ Ds + df \wedge Ds - df \wedge Ds = f\Theta_k s,$$

so that $\Theta_k s$ is a E-valued $(k+2)$-form. But even a little more is true.

1.5.25 Proposition. *There exists an* $\mathrm{Hom}(E, E)$*-valued* 2*-form* $\Theta(D)$ *such that*

$$D \circ Ds = s \wedge \Theta(D)$$

for any $k = 0, 1, ...$ *and any* E*-valued* k*-form* s*.*

Proof. We work in a local trivialization, where we denote by ω the connection matrix. We calculate that

$$\begin{aligned}\Theta_k s = D_{k+1} D_k s &= D_{k+1}(ds + (-1)^k s \wedge \omega) \\ &= d(ds + (-1)^k s \wedge \omega) + (-1)^{k+1}(ds + (-1)^k s \wedge \omega) \wedge \omega \\ &= (-1)^k(ds \wedge \omega + (-1)^k s \wedge d\omega) + (-1)^{k+1} ds \wedge \omega - (-1)^k s \wedge \omega \wedge \omega \\ &= s \wedge (d\omega - \omega \wedge \omega).\end{aligned}$$

Thus the k-independent local endomorphism

$$s \mapsto s \wedge (d\omega - \omega \wedge \omega)$$

agrees with $D \circ D$. Since $D \circ D$ is globally defined, the proposition is proved. □

Because $d\omega - \omega \wedge \omega$ is the localization of an endomorphism of E whose coefficients are 2-forms, we can expect a transformation rule.

1.5.26 Proposition. *If* g *is a change of frame and* $\tilde{\omega}$ *is the connection matrix in the new frame, then*

$$d\tilde{\omega} - \tilde{\omega} \wedge \tilde{\omega} = g^{-1}(d\omega - \omega \wedge \omega)g.$$

It is sometimes also useful to compute the action of the curvature on vector fields.

1.5.27 Proposition. *If* $\xi, \eta \in \Gamma(M, \mathscr{C}^\infty(T_M))$ *are smooth vector fields and* $s \in \Gamma(M, \mathscr{C}^\infty(E))$ *is a smooth section then*

$$\Theta(D)_{\eta,\xi} s = D_\xi D_\eta s - D_\eta D_\xi s - D_{[\xi,\eta]} s.$$

Proof. Fix a symmetric connection ∇ for T_M. We compute that

$$\begin{aligned}D_\xi D_\eta s - D_\eta D_\xi s &= D_\xi \langle Ds, \eta \rangle = \langle D \langle Ds, \eta \rangle, \xi \rangle - \langle D \langle Ds, \xi \rangle, \eta \rangle \\ &= \langle \langle DDs, \eta \rangle + \langle Ds, \nabla\eta \rangle, \xi \rangle - \langle \langle DDs, \xi \rangle - \langle Ds, \nabla\xi \rangle, \eta \rangle \\ &= \langle \langle DDs, \eta \rangle \xi \rangle - \langle \langle DDs, \xi \rangle, \eta \rangle + \left\langle Ds, \nabla_\xi \eta \right\rangle - \left\langle Ds, \nabla_\eta \xi \right\rangle \\ &= \Theta(D)_{\eta,\xi} s + \langle Ds, [\xi, \eta] \rangle,\end{aligned}$$

where the last equality follows from the symmetry of the connetion ∇. □

Curvature of the dual connection

Let $E \to M$ be a vector bundle with connection D_E. The dual connection is determined by the relation (1.8). Taking the exterior derivative of the latter, we have

$$\begin{aligned} 0 = d^2 \langle v, \xi \rangle &= d\left(\langle D_E v, \xi \rangle + \langle v, D_{E^\vee} \xi \rangle \right) \\ &= \langle D_E D_E v, \xi \rangle - \langle D_E v, D_{E^\vee} \xi \rangle + \langle D_E v, D_{E^\vee} \xi \rangle + \langle v, D_{E^\vee} D_{E^\vee} \xi \rangle \\ &= \langle D_E D_E v, \xi \rangle + \langle v, D_{E^\vee} D_{E^\vee} \xi \rangle , \end{aligned}$$

which proves the following proposition.

1.5.28 Proposition. *Let $E \to M$ be a vector bundle with connection D_E. Then*

$$\langle v, \Theta(D_{E^\vee}) \xi \rangle = - \langle \Theta(D_E) v, \xi \rangle .$$

Curvature of product connections

If $E \to M$ and $E_1 \to M$ are two vector bundles with connections D_E and D_{E_1} respectively then one has the product connection $D_{E \times E_1}$ for the vector bundle associated to whichever multilinear product $\times$ is under consideration. We compute that

$$\begin{aligned} & D_{E\times E_1} D_{E \times E_1} (v \times w) \\ = \; & D_{E \times E_1}((D_E v) \times w + v \times (D_{E_1} w)) \\ = \; & (D_E D_E v) \times w - (D_E v) \times D_{E_1} w) + (D_E v) \times (D_{E_1} w) + v \times (D_{E_1} D_{E_1} w). \end{aligned}$$

Since the two middle terms of the last line cancel each other, we have proved the following simple proposition.

1.5.29 Proposition. *The curvature $\Theta(D_{E\times E_1})$ of $D_{E \times E_1}$ satisfies*

$$\Theta(D_{E \times E_1})(v \times w) = (\Theta(D_E) v) \times w + v \times (\Theta(D_{E_1}) w).$$

Curvature of a metric compatible connection

1.5.30 Proposition. *Let $E \to M$ be a complex vector bundle with Hermitian metric h. If D is a connection for E such that $Dh = 0$ then $\Theta(D)$ is h-skew Hermitian, i.e.,*

$$h(\Theta(D)s, t) = -h(s, \Theta(D)t)$$

for all sections s, t of E.

Proof. We compute that

$$\begin{aligned} 0 = d^2 h(s,t) &= d(h(Ds,t) + h(s, Dt)) \\ &= h(DDs, t) - h(Ds, Dt) + h(Ds, Dt) + h(s, DDt) = h(DDs, t) + h(s, DDt), \end{aligned}$$

as claimed. □

Curvature of the Chern connection

In this section, fix a holomorphic Hermitian vector bundle $(E, h) \to X$. We want to compute the curvature of the unique holomorphic Hermitian connection, in terms of the metric. Before computing the curvature in a local frame, let us examine the Chern connection directly.

First observe that since $D = D^{1,0} + \bar\partial$ and $\bar\partial^2 = 0$,

$$D_1 \circ D = D_1^{1,0} \circ D^{1,0} + D_1^{1,0} \circ \bar\partial + \bar\partial_1 \circ D^{1,0}.$$

Thus far we have used the fact that the Chern connection is complex, but not that it is metric compatible. Metric compatibility reads as

$$\partial h(s,t) = h(D^{1,0}s, t) + h(s, \bar\partial t) \quad \text{and} \quad \bar\partial h(s,t) = h(\bar\partial s, t) + h(s, D^{1,0}t).$$

(Note that the first of these equations is consequence of the second via complex conjugation followed by interchange of the roles of s and t.) Since $\partial^2 = 0$,

$$0 = h(D_1^{1,0} \circ D^{1,0}s, t) + h(D^{1,0}s, \bar\partial t) - h(D^{1,0}s, \bar\partial t) + h(s, \bar\partial_1 \bar\partial t) = h(D_1^{1,0} \circ D^{1,0}s, t),$$

and thus $D_1^{1,0} \circ D^{1,0} = 0$. Therefore

$$\Theta = D_1^{1,0} \circ \bar\partial + \bar\partial_1 \circ D^{1,0}.$$

In particular, the curvature maps sections to twisted $(1,1)$-forms, and therefore the curvature form of the Chern connection is a $(1,1)$-form.

Let us compute the curvature form in a holomorphic frame $\{e_\alpha\}$ for E. Setting

$$h_{\alpha\bar\beta} = h(e_\alpha, e_\beta),$$

In the proof of Theorem 1.5.13 we saw that the connection matrix ω of $(1,0)$ forms is given by

$$\omega^\alpha_\beta = h^{\alpha\bar\mu} \partial h_{\beta\bar\mu}.$$

In matrix notation,

$$\omega = (\partial H) H^{-1}.$$

Using the formulas $\partial(H^{-1}) = -H^{-1}(\partial H)H^{-1}$ and $\bar\partial(H^{-1}) = -H^{-1}(\bar\partial H)H^{-1}$, we calculate that

$$\begin{aligned} d\omega - \omega \wedge \omega &= (\partial + \bar\partial)(\partial H H^{-1}) - \partial H H^{-1} \wedge \partial H H^{-1} \\ &= \bar\partial((\partial H)H^{-1}) + (\partial H)H^{-1}(\partial H)H^{-1} - (\partial H)H^{-1}(\partial H)H^{-1} = \bar\partial((\partial H)H^{-1}). \end{aligned}$$

We have therefore proved the following proposition.

1.5.31 Proposition. *Let $E \to X$ be a holomorphic vector bundle with Hermitian metric h and let $e_1, ..., e_r$ be a holomorphic frame over some open subset. The curvature of the Chern connection of $(E, h) \to X$ is given by the formula*

$$\Theta^\alpha_\beta = \bar\partial(h^{\alpha\bar\mu} \partial h_{\beta\bar\mu})$$

where $h_{\alpha\bar\beta} := h(e_\alpha, e_\beta)$ is the matrix of h in this frame and $h^{\alpha\bar\beta}$ is the matrix of the dual metric for E^* in the dual frame.

Curvature of determinant bundles

1.5.32 Proposition. *Let $(E, D_E) \to M$ be a vector bundle of rank r with connection, and let $(\det E, D_{\det E}) \to M$ be its determinant line bundle. Then*

$$\Theta(D_{\det E}) = \operatorname{trace}(\Theta(D_E)).$$

Proof. Let $e_1, ..., e_r$ be a frame for E. Then

$$D^2_{\det E}(e_1 \wedge ... \wedge e_r) = D_{\det E}\left(\sum_{j=1}^{r} e_1 \wedge ... \wedge D_E e_j \wedge ... \wedge e_r\right)$$

$$= \sum_{j=2}^{r}\sum_{k=1}^{j-1} e_1 \wedge ... \wedge D_E e_k \wedge ... \wedge D_E e_j \wedge ... \wedge e_r$$

$$- \sum_{j=1}^{r-1}\sum_{k=j+1}^{r} e_1 \wedge ... \wedge D_E e_j \wedge ... \wedge D_E e_k \wedge ... \wedge e_r + \sum_{j=1}^{r} e_1 \wedge ... \wedge D_E^2 e_j \wedge ... \wedge e_r$$

$$= \sum_{j=1}^{r} e_1 \wedge ... \wedge D_E^2 e_j \wedge ... \wedge e_r,$$

and thus $\Theta(D_{\det E}) = \operatorname{trace}(\Theta(D_E))$, as claimed. □

The canonical bundle

Given a complex manifold X of complex dimension n, the *canonical bundle* K_X of X is the line bundle $\det T_X^{*1,0}$. The local sections K_X are holomorphic n-forms.

If X is a Hermitian complex manifold with Hermitian metric h , theorem 1.5.32 tells us that the curvature of $(K_X, \det(h))$ is just the trace of the curvature of (X, h).

In the special case that the metric h is Kähler, this curvature is the negative of the so-called *Ricci curvature* of h:

$$\operatorname{Ricci}(h) = -\operatorname{trace}(\Theta(h)). \tag{1.16}$$

In the next paragraph, we will remind the reader of the definition, from Riemannian Geometry, of the Ricci curvature of a Riemannian metric. We will then show that (1.16) holds when the metric in question is Kähler.

1.5.7 Symmetry of Kähler curvature

Since the Levi-Civita connection is determined by vanishing of a certain part of the connection matrix, one can expect that less numerical data is needed to determine its curvature. In the Kähler case, where one has two vanishing conditions that have been required to hold simultaneously, one can expect even less numerical data to determine the curvature of the Kähler connection. This redundancy in the curvature matrices of the Levi-Civita and Kähler connections is expressed in terms of symmetries in the curvature tensor. In this section we write down some of the symmetries of the Kähler curvature.

Curvature of the Levi-Civita connection

Let (M, g) be a Riemannian manifold, and let ∇ be its Levi-Civita connection. As we have seen, ∇ is uniquely determined by the two conditions

$$d(g(\xi,\eta))(\zeta) = g(\nabla_\zeta \xi, \eta) + g(\xi, \nabla_\zeta \eta) \quad \text{and} \quad \nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta]$$

For all vector fields ξ, η and ζ on M. We shall now analyze what these symmetries mean for the curvature of the Levi-Civita connection.

It is customary to denote by R the curvature of the Levi-Civita connection. That is to say,

$$R(\xi,\eta)\zeta = \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]}\zeta.$$

The curvature tensor is locally defined to be

$$R_{ijk\ell} := g(R(\tfrac{\partial}{\partial x^k}, \tfrac{\partial}{\partial x^\ell})\tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^i}).$$

It is immediately clear that for any connection (Levi-Civita or not),

$$R_{ijk\ell} = -R_{ij\ell k}. \tag{1.17}$$

Since $d^2 = 0$, the curvature $\Theta(D)$ of any metric compatible connection D must satisfy

$$g(\Theta(D)\xi, \eta) + g(\xi, \Theta(D)\eta) = d^2(g(\xi,\eta) = 0.$$

This relation implies that

$$R_{ijk\ell} = -R_{jik\ell}. \tag{1.18}$$

The symmetry of the Levi-Civita connection implies that

$$\begin{aligned} R(\tfrac{\partial}{\partial x^k}, \tfrac{\partial}{\partial x^\ell})\tfrac{\partial}{\partial x^j} &= \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^\ell}} \tfrac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^\ell}} \nabla_{\frac{\partial}{\partial x^k}} \tfrac{\partial}{\partial x^j} \\ &= \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^j}} \tfrac{\partial}{\partial x^\ell} - \nabla_{\frac{\partial}{\partial x^\ell}} \nabla_{\frac{\partial}{\partial x^j}} \tfrac{\partial}{\partial x^k} \\ &= R(\tfrac{\partial}{\partial x^k}, \tfrac{\partial}{\partial x^j})\tfrac{\partial}{\partial x^\ell} - R(\tfrac{\partial}{\partial x^\ell}, \tfrac{\partial}{\partial x^j})\tfrac{\partial}{\partial x^k} + \nabla_{\frac{\partial}{\partial x^j}}(\nabla_{\frac{\partial}{\partial x^\ell}} \tfrac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^k}} \tfrac{\partial}{\partial x^\ell}) \\ &= -R(\tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^k})\tfrac{\partial}{\partial x^\ell} - R(\tfrac{\partial}{\partial x^\ell}, \tfrac{\partial}{\partial x^j})\tfrac{\partial}{\partial x^k}. \end{aligned}$$

Thus we obtain the *First Bianchi Identity*

$$R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0. \tag{1.19}$$

As a consequence, we have

$$\begin{aligned} &\tfrac{1}{2}(R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk}) = 0,\ \tfrac{1}{2}(R_{jk\ell i} + R_{jik\ell} + R_{j\ell ik}) = 0, \\ &\tfrac{1}{2}(R_{k\ell ij} + R_{kj\ell i} + R_{kij\ell}) = 0,\ \tfrac{1}{2}(R_{\ell ijk} + R_{\ell kij} + R_{\ell jki}) = 0. \end{aligned}$$

and adding these four equations and using (1.17) and (1.18) yields

$$R_{ik\ell j} = R_{\ell jik}. \tag{1.20}$$

Curvature of the Kähler connection

We agree to let Greek letters run through $\{1, ..., n\}$ and latin letters through $\{1, ..., n, \bar{1}, ..., \bar{n}\}$. We also use the notation

$$\partial_\alpha = \frac{\partial}{\partial z^\alpha} \quad \text{and} \quad \partial_{\bar{\alpha}} = \frac{\partial}{\partial \bar{z}^\alpha}, \quad 1 \leq \alpha \leq n.$$

Thus

$$\partial_i = \frac{\partial}{\partial z^i} \quad \text{and} \quad \partial_{\bar{i}} = \frac{\partial}{\partial \bar{z}^i} \qquad \text{for } i = 1, ..., n,$$

and

$$\partial_i = \frac{\partial}{\partial \bar{z}^i} \quad \text{and} \quad \partial_{\bar{i}} = \frac{\partial}{\partial z^i} \qquad \text{for } i = \bar{1}, ..., \bar{n}.$$

We denote by Γ^i_{jk} the Christoffel symbols of ∇, which are defined by the relation

$$\nabla_{\partial j} \partial_k = \Gamma^j_{ik} \partial_j.$$

Now let ∇ be the Kähler connection on a Kähler manifold (X, g). In Section 1.5.3 we presented the formula for the Christoffel symbols with respect to the real coordinates, and so they are different from the symbols we have defined in the present paragraph. Nevertheless, as already mentioned, since

$$s^{1,0}\tfrac{\partial}{\partial x^i} = -\sqrt{\text{-}1}s^{1,0}\tfrac{\partial}{\partial y^i} = \tfrac{\partial}{\partial z^i},$$

one does retain the symmetries of the Levi-Civita connection when passing from T_X to $T^{1,0}_X$.

Since ∇ is a the Connection of a Hermitian metric, by taking complex conjugates we see that

$$\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \overline{\Gamma^k_{ij}} \tag{1.21}$$

Next we observe that, since the Kähler connection matrix is a matrix of $(1,0)$-forms, its Christoffel symbols must satisfy

$$\Gamma^\alpha_{i\bar{\gamma}} = \Gamma^{\bar{\alpha}}_{i\gamma} = 0 \tag{1.22}$$

Let us introduce the following notation. Let R^j_i be the curvature tensor with respect to the local frame $\partial_1, ..., \partial_n, \partial_{\bar{1}}, ...\partial_{\bar{n}}$ for $T_X \otimes \mathbb{C}$ and write

$$R_{ij} = R^k_i g_{kj}.$$

The matrix entries R_{ij} are $(1,1)$-forms, and so we write

$$R_{ij} = R_{ijk\bar{\ell}} dz^k \wedge d\bar{z}^\ell.$$

(Note: this convention means $dz^{\bar{1}} = d\bar{z}^1$, etc.) Because the curvature is a $(1,1)$-form with skew-Hermitian symmetry, we have

$$R_{ij\alpha\beta} = R_{ij\bar{\alpha}\bar{\beta}} = 0 \quad \text{and} \quad R_{ij\bar{\alpha}\beta} = -R_{ij\beta\bar{\alpha}} = R_{ij\bar{\beta}\alpha}. \tag{1.23}$$

Now, since the Kähler connection is Chern, we have $(\nabla^{1,0})^2 = 0$ and $\bar\partial^2 = 0$, which reads as

$$R_{\alpha\beta ij} = R_{\bar\alpha\bar\beta ij} = 0. \tag{1.24}$$

Since the Kähler connection is also Levi-Civita, our work from the previous paragraph shows that

$$R_{ij\ell k} = R_{jik\ell} = -R_{k\ell ij}. \tag{1.25}$$

In particular, we have

$$R_{\alpha\bar\beta i\bar j} = R_{i\bar j\alpha\bar\beta} = R_{i\bar\beta\alpha\bar j}, \tag{1.26}$$

where the last equality is achieved as follows:

$$R_{i\bar j\alpha\bar\beta} = -R_{i\bar\beta\bar j\alpha} - R_{i\alpha\bar\beta\bar j} = -R_{i\bar\beta\bar j\alpha} = R_{i\bar\beta\alpha\bar j}.$$

Ricci curvature of the Kähler connection

Finally, we come to the promised discussion of Ricci curvature. Recall that for a Riemannian metric the Ricci curvature is the trace

$$\mathrm{Ricci}(g)_{ij} = g^{k\ell}R_{ik\ell j}.$$

As mentioned at the end of the previous paragraph, there is some symmetry to the Ricci curvature of a Kähler metric. Indeed, the Ricci curvature tensor satisfies

$$\mathrm{Ricci}(g)_{\alpha\beta} = \mathrm{Ricci}(g)_{\bar\alpha\bar\beta} = 0 \quad \text{and} \quad \mathrm{Ricci}(g)_{\alpha\bar\beta} = \overline{\mathrm{Ricci}(g)_{\beta\bar\alpha}}. \tag{1.27}$$

In fact, we claimed in the last paragraph that one has the formula

$$\mathrm{Ricci}(g)_{\alpha\bar\beta} = -\partial_\alpha\partial_{\bar\beta}\log\det\left(g_{\mu\bar\nu}\right), \tag{1.28}$$

where g is the Kähler metric in question. To see this formula, we use the above symmetries as follows.

$$-\mathrm{Ricci}(g)_{\alpha\bar\beta} := -g^{d\bar c}R_{\alpha\bar c d\bar\beta} \overset{(1.26)}{=} -g^{d\bar c}R_{\alpha\bar\beta d\bar c} \overset{(1.24)}{=} g^{d\bar c}R_{\alpha\bar\beta\bar c d} \overset{(*)}{=} g^{\delta\bar\gamma}R_{\alpha\bar\beta\bar\gamma\delta} = \partial_{z^\alpha}\partial_{\bar z^\beta}\log\det\left(g\right),$$

where $(*)$ holds because the metric is Hermitian, and so only has $(1,1)$-parts, and the last equality holds because, as we saw in Subsection 1.5.6,

$$\partial\bar\partial\log\det g = -\mathrm{trace}\ \bar\partial(\partial g g^{-1}).$$

1.5.8 Chern-Weil theory

Chern-Weil Theory deals with obtaining topological information from curvature of vector bundles. The topological information the theory provides is at the level of deRham cohomology, i.e., the theory produces closed forms that are not a priori exact. The theory has two steps: (i) produce closed forms from a given connection D for a vector bundle $E \to M$, and (ii) show that the cohomology classes of these forms in independent of the choice of connection D, and thus these classes define topological invariants of E.

At the heart of the first part of the theory is the Second Bianchi Identity (Theorem 1.5.33). The basic idea is to find functions $f(T)$ of certain types of tensors T that include curvature tensors of vector bundles, and for which the connection in question is 'compatible', in the sense that the chain rule

$$d(f(T)) = Df(T)(DT)$$

holds. In this manner, we will produce closed forms when we take $T = \Theta(D)$ (or for that matter, any tensor T satisfying $DT = 0$).

The second part of the theory, showing that these closed forms are independent of the choice of connection D, does not rely on the Second Bianchi Identity.

Since the curvature $\Theta(D)$ of a connection D of some vector bundle E is an $\mathrm{End}(E)$-valued 2-form, in any given frame $\Theta(D)$ is a matrix of 2-forms that transforms, after a change of frame, by conjugation with the transformation of the frames. It is therefore plausible that we consider only functions that are invariant under conjugation by elements of $GL(r, \mathbb{C})$, where $r = \mathrm{rank}(E)$. As we know from linear algebra, the coefficients of the characteristic polynomial of a matrix, as functions of the entries of the matrix, remain invariant under conjugation. These functions will play a major role in this section.

The Second Bianchi Identity

Let $E \to M$ be a smooth vector bundle. A connection for $E \to M$ induces a connection on $\mathrm{Hom}(E, E) = E \otimes E^*$, as well as exterior covariant derivatives for $\mathrm{Hom}(E, E)$-valued k-forms. The exterior covariant derivative of $\sigma \in \Gamma(X, \mathscr{C}^\infty(\Lambda^k(T^*_M) \otimes \mathrm{Hom}(E, E)))$ is denoted $D\sigma$.

1.5.33 Theorem. (Second Bianchi Identity) *If* $D : \Gamma(M, \mathscr{C}^\infty(E)) \to \Gamma(M, \mathscr{C}^\infty(T^*_M \otimes E))$ *is a connection then the curvature endomorphism* $\Theta = \Theta(D) \in \Gamma(X, \mathscr{C}^\infty(\Lambda^2(T^*_M) \otimes \mathrm{Hom}(E, E)))$ *of a connection* D *satisfies*

$$D\Theta = 0.$$

Proof. Let s be a local section of $E \to M$. Then on the one hand

$$D(DDs) = D(\Theta s) = (D\Theta)s + \Theta \wedge Ds,$$

and on the other hand

$$(DD)Ds = \Theta \wedge Ds.$$

therefore $(D\Theta)s = 0$ for all s, as claimed. □

Principal k-tensors

1.5.34 Definition. Let $E \to M$ be a vector bundle. A $\mathrm{Hom}(E,E)$-valued k-form, i.e., a section of $\mathrm{Hom}_k(E,E) := \Lambda^k(T^*_M) \otimes \mathrm{Hom}(E,E)$, is called a *principal k-tensor*.[1] The set of all principal k-tensors is denoted

$$\Gamma(M, \mathrm{Hom}_k(E,E)).$$

Fix a connection D for the vector bundle $E \to M$. There is an induced connection for $\mathrm{Hom}(E,E)$, and hence twisted exterior derivatives on principal k tensors. We denote all of these derivatives by D.

A local formula for D is given as follows. Let $e_1, ..., e_r$ be a frame for E, denote the dual frame for E^* by $e^{*1}, ..., e^{*r}$, and let $x^1, ..., x^m$ be local coordinates. In terms of these local data, $T \in \Gamma(M, \mathrm{Hom}_k(E,E))$ can be expressed as

$$T = T^i_{jI} dx^I \otimes e_i \otimes e^{*j}.$$

Then

$$\begin{aligned} DT &= \left((dT^i_{jI}) + T^\ell_{jI}\omega^i_\ell - T^i_{mI}\omega^m_j\right) \wedge dx^I \otimes e_i \otimes e^{*j} \\ &:= dT^i_{jI} \wedge dx^I \otimes e_i \otimes e^{*j} \\ &\qquad + (-1)^k \left(T^\ell_{jI} dx^I \wedge \omega^i_\ell \otimes e_i \otimes e^{*j} - (-1)^k \omega^m_j \wedge dx^I \otimes e_i \otimes (T^i_{mI} e^{*j})\right), \end{aligned}$$

where ω is the connection matrix for D in the frame $e_1, ..., e_r$ for E. Informally, we write

$$DT = dT + (-1)^k (T \wedge \omega - (-1)^k \omega \wedge T).$$

To make things even more concise, recall that for matrices of forms, one has a graded commutator: if A is a matrix of k-forms and B is a matrix of ℓ-forms then the graded commutator of A and B is

$$[A,B] = A \wedge B - (-1)^{k\ell} B \wedge A.$$

With this notation, our local formula rewrites as

$$DT = dT + (-1)^k [T, \omega].$$

The next result is an immediate consequence of the well-definedness of the higher exterior derivatives and of the induced connection associated to a product bundle.

1.5.35 Proposition.

1. $D : \Gamma(M, \mathrm{Hom}_k(E,E)) \to \Gamma(M, \mathrm{Hom}_{k+1}(E,E))$
2. $DDT = T \wedge \Theta - \Theta \wedge T = [T, \Theta]$.

[1] The name 'principal k-tensor' is not standard terminology.

Let r and j be positive integers, and denote by $M(r,\mathbb{C})$ the set of $r\times r$ matrices with $\mathbb{C}$ coefficients. We consider functions $P : M(r,\mathbb{C})^j \to \mathbb{C}$ such that

$$A^i \mapsto P(A^1, ..., A^i, ...A^j)$$

is linear for each $1 \le i \le j$. We shall call such functions *multi-linear*.

Multi-linear functions admit an action by $GL(r,\mathbb{C})$ and also by $\mathbb{S}_j$, the symmetric group on j letters, as follows. If $g \in GL(r,\mathbb{C})$ and $\sigma \in \mathbb{S}_j$, we set

$$g * P(A^1, ..., A^j) = P(gA^1g^{-1}, ..., gA^jg^{-1})$$

and

$$\sigma \cdot p(A^1, ..., A^j) = P(A^{\sigma(1)}, ..., A^{\sigma(j)}).$$

The functions that remain invariant under either of these actions have classical names.

1.5.36 Definition. A multi-linear function P that is invariant with respect to the $GL(r,\mathbb{C})$ action is called *invariant*, while one that is invariant with respect to the $\mathbb{S}_j$ action is called *symmetric.*

1.5.37 Example. If A is an $r\times r$ matrix with complex coefficients, we define homogeneous polynomials p_k of degree k $(1 \le k \le r)$ by the relation

$$\det\left(I + \frac{\sqrt{\text{-}1}}{2\pi}A\right) = 1 + \sum_{k=1}^{r} p_k(A).$$

We let $P_k(A^1, ...A^k)$ be the completely polarized polynomial of $p_k(A)$, with the normalization so chosen that $P_k(A, ..., A) = p_k(A)$. The formula for polarization is as follows: it is linear, and sends the monomial $a_{i_1j_1} \cdot ... \cdot a_{i_kj_k}$ to

$$\frac{1}{k!}\sum_{\sigma\in\mathbb{S}_k} a^{\sigma(1)}_{i_1j_1} \cdot ... \cdot a^{\sigma(k)}_{i_kj_k}.$$

The polynomials P_k are multilinear, invariant, and symmetric. Indeed, symmetry follows from polarization, invariance follows from polarization together with invariance properties of the characteristic polynomial of a matrix, and multilinearity follows from multilinearity properties for the coefficients of the characteristic polynomial. ⋄

1.5.38 Proposition. *Let $P : M(r,\mathbb{C})^j \to \mathbb{C}$ be a multilinear invariant symmetric function. If $A^k \in \Gamma(M, \mathrm{Hom}_{d_k}(E,E))$, $1 \le k \le j$, then*

$$d(P(A^1, ..., A^j)) = \sum_{\ell=1}^{j} P(A^1, ..., DA^\ell, ..., A^j). \tag{1.29}$$

Proof. The polynomials P may be viewed as sections of an appropriate trivial vector bundle. Since P is linear in each variable, Formula (1.29) is a consequence of the chain rule. □

If we specialize to the case of the "polarized characteristic polynomial" invariant symmetric functions $P = P_k$ and the curvature matrices with 2-form coefficients $A^\ell = \Theta(D)$, $1 \leq \ell \leq k$, then we obtain Part 1 of the following theorem.

1.5.39 THEOREM (Fundamental Theorem of Chern-Weil Theory). *Let $E \to M$ be a complex vector bundle of rank r with a connection having curvature form Θ.*

1. *The coefficients $P_j(\Theta)$ of the Chern polynomial*
$$\det\left(I + t\frac{\sqrt{-1}}{2\pi}\Theta\right) = 1 + \sum_{j=1}^{r} P_j(\Theta)t^j,$$
are closed $2j$-forms on M.

2. *For each j the cohomology class of $P_j(\Theta)$ is independent of the connection on $E \to M$.*

1.5.40 DEFINITION. With the notation of theorem 1.5.39, the cohomology class

$$c_i(E) := [P_i(\Theta)] \in H^{2i}(M, \mathbb{C}), \quad 1 \leq i \leq r$$

is called the i^{th} Chern class of E.

Proof of part 2 of Theorem 1.5.39. Let ω_0 and ω_1 be the connection matrices for the connections D_0 and D_1 on E. By Proposition 1.5.3) the difference $\eta = \omega_1 - \omega_0$ of two connection matrices is the matrix of an element of $\Gamma(M, \mathrm{Hom}_1(E, E))$.

Now consider the connections D_t whose connection matrices are $\omega_t = \omega_0 + t\eta$, for $0 \leq t \leq 1$. The curvature of D_t is

$$\begin{aligned}\Theta_t &= d\omega_t - \omega_t \wedge \omega_t = d\omega_0 + t(d\eta - \omega_0 \wedge \eta - \eta \wedge \omega_0) - \omega_0 \wedge \omega_0 - t^2 \eta \wedge \eta \\ &= \Theta_0 + tD_0\eta - t^2\eta \wedge \eta.\end{aligned}$$

We note that

$$\frac{d\Theta_t}{dt} = D_0\eta - 2t\eta \wedge \eta = D_t\eta. \tag{1.30}$$

Let $P(A^1, ..., A^j)$ be an invariant symmetric polynomial and set

$$P(A) = P(A, ..., A) \quad \text{and} \quad Q(B, A) = jP(B, \underbrace{A, ..., A}_{j-1})$$

Then by the symmetry of P, (1.30), Proposition 1.5.38 and the Second Bianchi Identity (Theorem 1.5.33) we have

$$\frac{d}{dt}P(\Theta_t) = Q(D_t\eta, \Theta_t) = d(Q(\eta, \Theta_t)).$$

But then

$$P(\Theta_1) - P(\Theta_0) = d\left(\int_0^1 Q(\eta, \Theta_t)dt\right).$$

Taking $P = P_j$ completes the proof. $\square$

Chapter 2

Two Fundamental PDE

2.1 Some Functional Analysis

2.1.1 Densely defined operators with closed graphs

The $\bar{\partial}$-equation defines an unbounded but nevertheless well-behaved operator on certain natural Hilbert spaces. Our first goal is to discuss the general class of such operators.

Let H_1 and H_2 be Hilbert spaces. We are interested in linear operators from subspaces of H_1 into H_2. Each such operator $T : H_1 \to H_2$ comes with its own domain Domain(T) which need not be all of H_1, and T need not be continuous on its domain.
It may sometimes be possible to extend an operator T to a larger domain. If this is so, and S is such an extension, then the graph

$$\mathrm{Graph}(T) := \{(x, Tx) \ ; \ x \in \mathrm{Domain}(T)\} \subset H_1 \times H_2$$

of T is a subspace of the graph Graph(S) of S.

2.1.1 Definition. An operator $T : H_1 \to H_2$ is said to be

(i) densely defined if Domain(T) is a dense subspace of H_1, and

(ii) closed if Graph(T) is a closed subspace of $H_1 \times H_2$.

2.1.2 Remark. If an operator is bounded on a Hilbert space (or more generally on a Banach space) then the Closed Graph Theorem tells us that it is closed. ◇

As the reader will recall, a Hilbert space is isometrically isomorphic to its dual. Indeed, the content of the Riesz Representation Theorem is that the conjugate linear map sending $v \in H$ to the bounded linear functional $\lambda_v \in H^*$ define by

$$\lambda_v w := \langle w, v \rangle$$

is a conjugate linear isometric isomorphism of the Hilbert spaces H and H^*.

Given a linear operator $T : H_1 \to H_2$ defined on a not-necessarily dense subspace of H_1, one can define its adjoint $T^* : H_2 \to H_1$ in a canonical way as follows.

a. The domain of T^* consists of all $\eta \in H_2$ such that

$$\mathscr{L}_\eta : x \mapsto \langle Tx, \eta \rangle_2$$

is a continuous linear functional on Domain(T).

b. If we then extend $\mathscr{L}_\eta$ to all of H_1 by defining $\mathscr{L}_\eta(y) = 0$ for all $y \perp$ Domain(T) then the Riesz Representation Theorem tells us that $\mathscr{L}_\eta$ is represented by a unique element $T^*\eta$ of H_1 satisfying

$$||T^*\eta||_1 = ||\mathscr{L}_\eta|| \quad \text{and} \quad \langle Tx, \eta \rangle_2 = \langle x, T^*\eta \rangle_1 \text{ for all } x \in \text{Domain}(T).$$

The linearity of $\eta \mapsto T^*\eta$ is easy to prove.

2.1.3 PROPOSITION. *Let $T : H_1 \to H_2$ be a linear operator.*

i. *If T is densely defined then $T^* : H_2 \to H_1$ is closed.*

ii. *If T is closed then T^* is densely defined.*

In particular, if T is closed and densely defined then so is T^*.

Proof. Let $F : H_1 \times H_2 \to H_1 \times H_2$ be given by $F(\xi, \eta) = (-\xi, \eta)$. Then $(\xi, \eta) \perp F(\text{Graph}(T))$, i.e.,

$$\langle x, \xi \rangle_1 = \langle Tx, \eta \rangle_2 \qquad \text{for all } x \in \text{Domain}(T),$$

if and only if

$$\eta \in \text{Domain}(T^*) \quad \text{and} \quad \langle x, T^*\eta - \xi \rangle = 0 \text{ for all } x \in \text{Domain}(T).$$

Therefore

$$F(\text{Graph}(T))^\perp = \text{Graph}(T^*) \oplus (\text{Domain}(T)^\perp \times \{0\}). \tag{2.1}$$

Thus if T is densely defined, i.e., Domain$(T)^\perp = 0$, then T^* is closed (which proves i).

Next let $\eta \in \text{Domain}(T^*)^\perp$. Then $(0, \eta) \perp \text{Graph}(T^*)^\perp \oplus (A \times \{0\})$ for any subspace $A \subset H_1$. Therefore from (2.1) we deduce that $(0, \eta) \perp F(\text{Graph}(T))^\perp$, i.e.,

$$(0, \eta) \in F(\text{Graph}(T))^{\perp\perp} = \overline{F(\text{Graph}(T)} = F(\overline{\text{Graph}(T)}).$$

Thus if T is closed and $\eta \in \text{Domain}(T^*)^\perp$ then $(-0, \eta) \in \text{Graph}(T)$, and hence $\eta = T0 = 0$. That is to say, T^* is densely defined, and the proof is complete. □

2.1.4 PROPOSITION. *If T is a closed, densely defined operator then $T^{**} = T$.*

Proof. By construction of T^*, if $x \in \text{Domain}(T)$ and $\eta \in \text{Domain}(T^*)$ then $\langle T^*\eta, x\rangle = \langle \eta, Tx\rangle$. By Proposition 2.1.3 the adjoint T^* is also closed and densely defined. Since $\eta \mapsto \langle \eta, Tx\rangle$ is continuous, $\text{Domain}(T) \subset \text{Domain}(T^{**})$.

Let $\theta \in \text{Domain}(T^{**})$. Since T is densely defined there is a sequence $\{x_j\} \subset \text{Domain}(T)$ such that $x_j \to \theta$. Then for all $\eta \in \text{Domain}(T^*)$ we have

$$\langle T^{**}\theta, \eta\rangle = \langle \theta, T^*\eta\rangle = \lim \left\langle x_j, T^*\eta \right\rangle = \lim \left\langle Tx_j, \eta \right\rangle .$$

Thus $Tx_j \to T^{**}\theta$ in H_2, and therefore $\text{Graph}(T) \ni (x_j, Tx_j) \to (\theta, T^{**}\theta)$ in $H_1 \times H_2$. Since T is closed, $(\theta, T^{**}\theta) \in \text{Graph}(T)$, and thus $\theta \in \text{Domain}(T)$ and $T^{**}\theta = T\theta$. □

2.1.5 COROLLARY. *If $T : H_1 \to H_2$ is a closed and densely defined linear operator then*

$$\text{Kernel}(T)^\perp = \overline{\text{Image}(T^*)}.$$

Proof. Suppose $y \in \overline{\text{Image}(T^*)}$. Then there exist $\text{Image}(T^*) \ni T^*\eta_j = y_j \to y$, and hence for all

$$\langle x, y\rangle = \lim \left\langle x, T^*\eta_j \right\rangle = \left\langle Tx, \eta_j \right\rangle = 0 \quad \text{ for all } x \in \text{Kernel}(T),$$

i.e., $y \in \text{Kernel}(T)^\perp$.

Next suppose $y \notin \overline{\text{Image}(T^*)}$. Then there exists $w \in \text{Image}(T^*)^\perp$ such that $\langle w, y\rangle \neq 0$. Since $w \in \text{Image}(T^*)^\perp$, i.e., $\langle w, T^*\eta\rangle = 0$ for all $\eta \in \text{Domain}(T^*)$ we have

(i) $w \in \text{Domain}(T^{**}) = \text{Domain}(T)$, and

(ii) $\langle Tw, \eta\rangle = 0$ for all $\eta \in \text{Domain}(T^*)$.

Since T^* is densely defined, we conclude that $Tw = 0$, i.e., that $w \notin \text{Kernel}(T)^\perp$. □

2.1.2 The Functional Analysis Lemma

Let $T : H_1 \to H_2$ be a closed, densely defined operator. If $\alpha \in H_2$ is in the image of T, say $\alpha = Tu$ form some $u \in H_1$, then for any $\beta \in \text{Domain}(T^*)$

$$\sup_{\beta \in \text{Domain}(T^*)} \frac{|(\alpha, \beta)|^2}{||T^*\beta||^2} = \sup_{\beta \in \text{Domain}(T^*)} \frac{|(u, T^*\beta)|^2}{||T^*\beta||^2} \leq ||u||^2.$$

Let u_o be the projection of u onto the closure of $\text{Image}(T^*)$. Then $u - u_o = \lim v_j$ for some sequence $\{v_j\} \subset \text{Image}(T^*)^\perp$. Thus for all $\beta \in \text{Domain}(T^*)$

$$(T(u - u_o), \beta) = (u - u_o, T^*\beta) = \lim(v_j, T^*\beta) = 0.$$

Since the domain of T^* is dense, we find in particular that

$$Tu_o = \alpha \quad \text{and} \quad \sup_{\beta \in \text{Domain}(T^*)} \frac{|(\alpha, \beta)|^2}{||T^*\beta||^2} = ||u_o||^2..$$

The next lemma essentially says that the converse is true.

2.1.6 LEMMA (Functional Analysis Lemma). *Let $T : H_1 \to H_2$ be a closed, densely defined operator and let $\alpha \in H_2$. Suppose there is a constant $C > 0$ such that*

$$|(\alpha, \beta)|^2 \leq C||T^*\beta||^2 \quad \textit{for all } \beta \in \text{Domain}(T^*). \tag{2.2}$$

Then there exists $u \in H_1$ such that

$$Tu = \alpha \quad \textit{and} \quad ||u||^2 \leq C.$$

Proof. Consider the linear functional

$$\mathscr{L} : \text{Image}(T^*) \ni T^*\beta \mapsto (\beta, \alpha).$$

By (2.2), if $T^*\beta = 0$ then $\beta \perp \alpha$ and thus $\mathscr{L}$ is well-defined and continuous. Defining $\mathscr{L}$ to vanish on $\text{Image}(T^*)^\perp$ extends $\mathscr{L}$ continuously to a linear function on H_1, and moreover $||\mathscr{L}||^2 \leq C$. By the Riesz Representation Theorem there exists $u \in H_1$ such that

$$\mathscr{L}(v) = (u, v) \text{ for all } v \in H_1 \quad \text{and} \quad ||u||^2 = ||\mathscr{L}||^2 \leq C.$$

Restricting to $\text{Image}(T^*)$, we have

$$(u, T^*\beta) = \mathscr{L}(T^*\beta) = (\alpha, \beta) \quad \text{for all } \beta \in \text{Domain}(T^*).$$

The Cauchy-Schwarz Inequality shows that $u \in \text{Domain}(T^{**})$. Since $T^{**} = T$ and $\text{Domain}(T^*)$ is dense, $Tu = \alpha$. The proof is complete. □

2.1.7 COROLLARY. *Let $T : H_1 \to H_2$ be a closed, densely defined operator. Suppose there exists $c > 0$ such that*

$$||\beta||_2^2 \leq c||T^*\beta||_1^2 \quad \textit{for all } \beta \in \text{Domain}(T^*). \tag{2.3}$$

Then T is surjective and there exists a bounded linear operator $\sigma : H_2 \to H_1$ such that

$$T \circ \sigma = \text{Id}_{H_2} \quad \textit{and} \quad ||\sigma||^2 \leq c.$$

Proof. Let $\alpha \in H_2$. Then for all $\beta \in H_2$ we have

$$|(\alpha, \beta)| \leq ||\alpha|| \cdot ||\beta|| \leq c||\alpha|| \cdot ||T^*\beta||.$$

By Lemma 2.1.6 there exists $u \in H_1$ such that $Tu = \alpha$ and $||u||^2 \leq c||\alpha||^2$. In particular, T is surjective. Define

$$\sigma(\alpha) := u_o$$

where u_o is the solution of $Tu = \alpha$ having minimal norm. Then $||\sigma(\alpha)||^2 \leq c||\alpha||^2$. Thus σ is bounded and $||\sigma||^2 \leq c$, as claimed. □

Remark on the condition (2.3)

Let $F : H_1 \to H_2$ be a closed, densely defined linear operator. We claim that the inequality

(2.4) $$||w||^2 \le C||F^*w||^2 \quad \text{for all } w \in \text{Domain}(F^*) \cap \text{Image}(F)$$

holds if and only if the image of F is a closed subspace of H_2.

Proof. Upon replacing H_2 be the closure of the image of F, we may assume that the image of F is dense.

Suppose then that (2.4) holds. Then by continuity (2.4) holds for all $w \in \text{Domain}(F^*)$, and hence by Corollary 2.1.7 F is surjective.

Conversely, suppose the image of F is closed, i.e., F is surjective. Let $\beta \in \text{Domain}(F^*)$. Then there exists $x \in \text{Domain}(T)$ such that $\beta = Fx$. We therefore have for each $\beta \in \text{Domain}(F^*)$ there exists

$$|(Fx, \beta)|^2 = |(x, F^*\beta)|^2 \le ||F^*\beta||^2||x||^2$$

Consider the set

$$\mathscr{A} := \{T_\beta := (\cdot, \beta)/||F^*\beta|| \ ; \ \beta \in \text{Domain}(F^*) \text{ and } F^*\beta \neq 0\} \subset H_2^*.$$

The above inequality says that for each $y \in H_2$

$$\sup_{T_\beta \in \mathscr{A}} |T_\beta(y)| \le ||x||,$$

where $y = Fx$. By the Uniform Boundedness Principle

$$C := \sup_{T_\beta \in \mathscr{A}} ||T_\beta||^2 < +\infty.$$

That is to say,

$$||\beta||^2 = ||F^*\beta||^2||T_\beta||^2 \le C||F^*\beta||^2,$$

which is the inequality we seek. □

2.2 The Bochner-Kodaira-Nakano Formula

2.2.1 Eliminating the volume form

Let X be a complex manifold with Hermitian metric g and let $E' \to X$ be a holomorphic vector bundle with Hermitian metric h. Since

$$\Lambda_X^{p,q} \otimes E' \cong \Lambda_X^{n,q} \otimes \left(\Lambda_X^{p,0} \otimes K_X^* \otimes E'\right),$$

(p,q)-form with values in E' can seen as (n,q)-forms with values in the holomorphic vector bundle $E := \Lambda_X^{p,0} \otimes K_X^* \otimes E'$. The Hermitian metrics g and h' induce the metric $h := g^{\otimes p} \otimes (\det g)^* \otimes h'$ on E. We work with E and h, and therefore always assume that $p = n$.

To a pair of E-valued (n,q)-forms φ, ψ we can associate the complex smooth measure $\langle \varphi, \psi \rangle_{g,h}$ obtained by pairing the (n,n)-form with values in the real vector bundle $E \otimes E^{\dagger} \to X$. Since the metric h can be viewed as a section of the real vector bundle $E^* \otimes E^{*\dagger} \to X$, pairing $\frac{\sqrt{-1}^{n^2}}{2^n} \varphi \wedge \bar{\psi}$ with h results in a complex measure, we we denote

$$\langle \varphi, \psi \rangle_{g,h}.$$

If we choose coordinates z and a frame $e_1, ..., e_r$, we can write

$$\varphi = \varphi^{\alpha}_{\bar{J}} dz^1 \wedge ... \wedge dz^n \wedge d\bar{z}^J \otimes e_{\alpha} \quad \text{and} \quad \psi = \psi^{\alpha}_{\bar{J}} dz^1 \wedge ... \wedge dz^n \wedge d\bar{z}^J \otimes e_{\alpha},$$

the resulting complex measure can be written as

$$\langle \varphi, \psi \rangle_{g,h} = h_{\alpha\bar{\beta}} \varphi^{\alpha}_{\bar{J}} \overline{\psi^{\beta}_{\bar{I}}} g^{I\bar{J}} dV(z),$$

where $g^{I\bar{J}} = g^{i_1\bar{j}_1} \cdot ... \cdot g^{i_q\bar{j}_q}$ where $dV(z)$ denotes Lebesgue measure in the coordinate chart.

2.2.2 Hilbert spaces

Let (X, g) be a Hermitian manifold of complex dimension n and let $E \to X$ be a holomorphic vector bundle with Hermitian metric h. We define the L^2 inner product of two smooth E-valued (n,q)-forms φ and ψ to be

$$(\varphi, \psi) := \int_X \langle \varphi, \psi \rangle_{g,h}. \tag{2.5}$$

2.2.1 Definition. We denote by $L^2_{n,q}(g,h)$ the Hilbert space closure of $\Gamma(X, \mathscr{C}^{\infty}(\Lambda^{p,q}_X \otimes E))$ with respect to the inner product (2.5). ◇

We shall also be naturally lead to study $(0,s)$-forms with values in the non-holomorphic vector bundle $\Lambda^{n,q}_X \otimes E$. In local coordinates z and with respect to a holomorphic frame $a_1, ..., e_r$ for E, such a form has the expression

$$\xi = \xi^{\alpha}_{\bar{\mu},\bar{J}} d\bar{z}^S \otimes \mathbf{dz} \wedge d\bar{z}^J \otimes e_{\alpha},$$

where $\mu = (\mu^1, ..., \mu^s)$ and $J = (j_1, ..., j_q)$ are multiindices, and

$$\mathbf{dz} := dz^1 \wedge ... \wedge dz^n.$$

As with the case of forms, we can define a complex measure, this time given by the formula

$$\{\xi, \eta\}_{g,h} := h_{\alpha\bar{\beta}} g^{\mu\bar{\nu}} g^{I\bar{J}} \xi^{\alpha}_{\bar{\nu},\bar{J}} \overline{\eta^{\beta}_{\bar{\mu},\bar{I}}} dV(z),$$

and we denote the corresponging L^2 inner product by

$$\{\!\{\xi, \eta\}\!\} := \frac{1}{s!} \int_X \{\xi, \eta\}_{g,h}.$$

In the case $s = 0$, this inner product agrees with (2.5). We shall not need the Hilbert space completion of the higher-order smooth forms.

2.2.3 $\bar\partial$ and its formal adjoint

Sections of the vector bundle $\Lambda_X^{p,q}\otimes E = \Lambda_X^{0,q}\otimes(\Lambda_X^{p,0}\otimes E)$ can be seen as $(0,q)$-forms with values in the *holomorphic* vector bundle $\Lambda^{p,0}\otimes E$, and thus we have a $\bar\partial$-operator acting on such sections. We denote by D the Chern connection associated to the Hermitian holomorphic vector bundle (E,h). The following proposition computes the $\bar\partial$ operator in terms of this Chern connection.

2.2.2 Proposition. *Let $\varphi = \varphi^\alpha_{\bar J_q}\mathbf{dz}\wedge d\bar z^{J_q}\otimes e_\alpha$ be a E-valued (n,q)-form. Then*

$$\left(\bar\partial\varphi\right)^\alpha_{\bar j_0\bar j_1\dots\bar j_q} = (-1)^n\sum_{k=0}^{q}(-1)^k(D_{\bar j_k}\varphi)^\alpha_{\bar j_0\dots\hat{\bar j}_k\dots\bar j_q}.$$

Proof. By the definition of $\bar\partial$

$$\begin{aligned}\bar\partial\varphi &= (-1)^n\partial_{\bar j}\varphi^\alpha_{\bar j_0\dots\bar j_{q-1}}\mathbf{dz}\wedge d\bar z^{j}\wedge d\bar z^{j_0}\wedge\dots\wedge d\bar z^{j_{q-1}}\otimes e_\alpha\\ &= (-1)^n\sum_{k=0}^{q}(-1)^k\partial_{\bar j_k}\varphi^\alpha_{\bar j_0\dots\hat{\bar j}_k\dots\bar j_q}\mathbf{dz}\wedge d\bar z^{j_0}\wedge\dots\wedge d\bar z^{j_q}\otimes e_\alpha\\ &= (-1)^n\sum_{k=0}^{q}(-1)^k(D_{\bar j_k}\varphi)^\alpha_{\bar j_0\dots\hat{\bar j}_k\dots\bar j_q}\mathbf{dz}\wedge d\bar z^{j_0}\wedge\dots\wedge d\bar z^{j_q}\otimes e_\alpha,\end{aligned}$$

where the last equality holds since $D^{0,1}=\bar\partial$ for the Chern connection. □

2.2.3 Definition. Let $T: L^2_{p,q}(g,h)\to L^2_{p',q'}(g',h')$ be a densely defined operator whose domain contains all smooth, compactly supported E-valued (p,q)-forms. The formal adjoint of T is the operator

$$T^*:\Gamma(X,\mathscr{C}^\infty(\Lambda_X^{p',q'}\otimes E))\to\Gamma(X,\mathscr{C}^\infty(\Lambda_X^{p,q}\otimes E))$$

defined by the requirement that

$$(T^*\varphi,\psi)' = (\varphi, T\psi)$$

for all $\varphi\in\Gamma(X,\mathscr{C}^\infty(\Lambda_X^{p',q'}\otimes E))$ and all compactly supported $\psi\in\Gamma(X,\mathscr{C}^\infty(\Lambda_X^{p,q}\otimes E))$.

The next proposition computes the formal adjoint of $\bar\partial$.

2.2.4 Definition. We define the operator $\tau_g:\Gamma(X,\mathscr{C}^\infty(\Lambda_X^{n,q}\otimes E))\to\Gamma(X,\mathscr{C}^\infty(\Lambda_X^{n,q-1}\otimes E))$

$$\tau_g\varphi := \left(\Lambda_g\partial\omega_g\right)^{\sharp_g}\varphi,$$

where Λ_g is the metric trace, i.e., the dual of the Lefschetz operator $\varphi\mapsto\omega_g\wedge\varphi$.

Remark. If the metric g is Kähler then by the symplectic formulation of the Kähler condition the operator τ_g is identically zero. ◇

2.2.5 Proposition. *Let $\varphi = \varphi^{\alpha}_{\bar{J}_q} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^{J_q} \otimes e_\alpha$ be a E-valued (n,q)-form. Then*

$$\bar{\partial}^* \varphi = (-1)^{n+1} g^{i\bar{j}} (D_i \varphi)^{\alpha}_{\bar{j}\bar{J}} \mathbf{dz} \wedge d\bar{z}^J \otimes e_\alpha - \tau_g \varphi.$$

Proof. If ψ is an E-valued $(n,q-1)$-form with compact support then

$$\begin{aligned}
&(\bar{\partial}^*\varphi, \psi) = (\varphi, \bar{\partial}\psi) \\
&= \frac{1}{n!q!} \int_X \varphi^{\alpha}_{\bar{j}_1 \dots \bar{j}_q} (-1)^n \sum_{k=1}^{q} (-1)^{k+1} \overline{(D_{\bar{j}'_k}\psi)^{\beta}_{\bar{j}'_1 \dots \widehat{\bar{j}'_k} \dots \bar{j}'_q}} g^{J'_q \overline{J_q}} h_{\alpha\bar{\beta}} \\
&= \frac{1}{n!q!} \int_X (-1)^{n+1} \sum_{k=1}^{q} (-1)^{k+1} \left(g^{j'_k \bar{j}_k} (D_{j'_k}\varphi)^{\alpha}_{\bar{j}_1 \dots \bar{j}_q} + \frac{\partial g^{j'_k \bar{j}_k}}{\partial z^{j'_k}} \varphi^{\alpha}_{\bar{j}_1 \dots \bar{j}_q} \right) \overline{\psi^{\beta}_{\bar{j}'_1 \dots \widehat{\bar{j}'_k} \dots \bar{j}'_q}} g^{j'_1 \bar{j}_1} \dots \widehat{g^{j'_k \bar{j}_k}} \dots g^{j'_q \bar{j}_q} h_{\alpha\bar{\beta}} \\
&= \frac{1}{n!(q-1)!} \int_X (-1)^{n+1} \left(g^{i\bar{j}} (D_i\varphi)^{\alpha}_{\bar{j}\bar{j}_1 \dots \bar{j}_{q-1}} - (-1)^q \frac{\partial g^{i\bar{j}}}{\partial z^i} \varphi^{\alpha}_{\bar{j}\bar{j}_1 \dots \bar{j}_{q-1}} \right) \overline{\psi^{\beta}_{\bar{j}'_1 \dots \bar{j}'_{q-1}}} g^{J'_{q-1} \overline{J_{q-1}}} h_{\alpha\bar{\beta}},
\end{aligned}$$

where the third equality follows from the metric compatibility of the Chern connection. From the relation $\delta^{\bar{j}}_{\bar{\ell}} = g^{i\bar{j}} g_{i\bar{\ell}}$ and its derivative we have

$$\frac{\partial g^{i\bar{j}}}{\partial z^j} = g^{k\bar{\ell}} g_{i\bar{\ell}} \frac{\partial g^{i\bar{j}}}{\partial z^k} = -g^{k\bar{\ell}} g^{i\bar{j}} \frac{\partial g_{i\bar{\ell}}}{\partial z^k},$$

and the result follows from the definition of τ_g. □

2.2.4 Another $\bar{\partial}$ operator

We can view E-valued (n,q)-forms as sections of the vector bundle $\Lambda^{n,q}_X \otimes E \to X$, ignoring the differential form nature of these sections. As we mentioned in Paragraph 2.2.2, this vector bundle is not holomorphic, so it does not have a canonical $\bar{\partial}$-operator. However, the Hermitian metric g for X yields the canonical isomorphism

$$\mathfrak{g} : \Lambda^{n,q}_X \otimes E \to \Lambda^{n,0}_X \otimes (\Lambda^q(T^{1,0}_X)) \otimes E;$$

the isomorphism is the one induced by the map $f \mapsto f^{\sharp_g}$ which sends a $(0,1)$-form f to the $(1,0)$-vector field $f^{\sharp_g}$ defined by

$$f^{\sharp_g} \lrcorner \omega_g = f,$$

or equivalently,

$$\left\langle f, \bar{\xi} \right\rangle = g(f^{\sharp_g}, \bar{\xi}) \quad \text{for all } \xi \in T^{1,0}_X.$$

We can extend the isomorphism $\mathfrak{g}$ to an isomorphism

$$\mathfrak{g} : \Lambda^{0,s}_X \otimes \left(\Lambda^{n,q}_X \otimes E\right) \to \Lambda^{0,s+1}_X \otimes \left(\Lambda^{n,0}_X \otimes (\Lambda^q(T^{1,0}_X)) \otimes E\right);$$

by acting on the second factor, i.e., the components of the $\Lambda^{n,q}_X \otimes E$-valued $(0,s)$-form.

Since the vector bundle $\Lambda_X^{n,0} \otimes (\Lambda^q(T_X^{1,0})) \otimes E$ *is* holomorphic, it has a $\bar\partial$-operator. We can conjugate the latter to the operator

$$\bar\nabla := \mathfrak{g}^{-1} \circ \bar\partial \circ \mathfrak{g} \quad \text{on } (\Lambda_X^{n,q} \otimes E)\text{-valued } (0,s)\text{-forms.}$$

In the case $s = 1$ we have the local formula

$$\begin{aligned}
\bar\nabla\varphi &= g_{I\bar L}\bar\partial\left(g^{I\bar J}\varphi^\alpha_{\bar J}\right) \otimes \mathbf{dz} \wedge d\bar z^L \otimes e_\alpha \\
&= \bar\partial(\varphi^\alpha_{\bar J}) \otimes \mathbf{dz} \wedge d\bar z^J \otimes e_\alpha - g^{I\bar J}\varphi^\alpha_{\bar J}\bar\partial(g_{I\bar L}) \otimes \mathbf{dz} \wedge d\bar z^L \otimes e_\alpha \\
&= \bar\partial(\varphi^\alpha_{\bar J}) \otimes \mathbf{dz} \wedge d\bar z^J \otimes e_\alpha - \left(\sum_{\nu=1}^q g^{i_\nu \bar j}\bar\partial(g_{i_\nu\bar\ell_\nu})\varphi^\alpha_{\bar\ell_1\ldots(\bar j)_\nu\ldots\bar\ell_q}\right) \otimes \mathbf{dz} \otimes d\bar z^L \otimes e_\alpha \\
&= \bar\partial(\varphi^\alpha_{\bar J}) \otimes \mathbf{dz} \wedge d\bar z^J \otimes e_\alpha - \left(\sum_{\nu=1}^q g^{i_\nu \bar j}\bar\partial(g_{i_\nu\bar\ell_1})\varphi^\alpha_{\bar j\bar\ell_2\ldots\bar\ell_q}\right) \otimes \mathbf{dz} \otimes d\bar z^L \otimes e_\alpha \\
&= \bar\partial(\varphi^\alpha_{\bar J}) \otimes \mathbf{dz} \wedge d\bar z^J \otimes e_\alpha - q\left(g^{i\bar j}\bar\partial(g_{i\bar\ell_1})\varphi^\alpha_{\bar j\bar\ell_2\ldots\bar\ell_q}\right) \otimes \mathbf{dz} \otimes d\bar z^L \otimes e_\alpha,
\end{aligned}$$

or equivalently,

$$\bar\nabla\varphi = \frac{\partial\varphi^\alpha_{\bar J}}{\partial\bar z^k} d\bar z^k \otimes \mathbf{dz} \wedge d\bar z^J \otimes e_\alpha - q\left(g^{i\bar j}\frac{\partial g_{i\bar\ell_1}}{\partial\bar z^k}\varphi_{\bar j\bar\ell_2\ldots\bar\ell_q}\right) d\bar z^k \otimes \mathbf{dz} \otimes d\bar z^L \otimes e_\alpha.$$

Since the operator $\bar\nabla$ satisfies $\bar\nabla^2 = 0$ and the vector bundle $\Lambda_X^{n,q} \otimes E$ is equipped with a Hermitian metric $\mathfrak{h}$ induced by h and g, there is a Chern connection for the triple $(\Lambda_X^{n,q} \otimes E, \bar\nabla, \mathfrak{h})$. We denote this Chern connection by ∇; as usual, it is defined by $\nabla = \nabla^{1,0} + \bar\nabla$, where

$$\mathfrak{h}(\nabla^{1,0}\sigma, s) := \partial\mathfrak{h}(\sigma, s) - \mathfrak{h}(\sigma, \bar\nabla s) \quad \text{for all smooth sections } \sigma, s.$$

The next proposition computes the formal adjoint $\bar\nabla^*$.

2.2.6 Proposition. *Let ξ be a smooth $(0,1)$-form with values in $\Lambda_X^{n,q} \otimes E$. Then*

$$\bar\nabla^*\xi = -\mathrm{Trace}(\mathbf{D}^{1,0}\mathfrak{g}\xi) - \tau_g\xi, \tag{2.6}$$

where $\mathbf{D}$ is the twisted exterior derivative associated to the Chern connection for $T_X^{1,0} \otimes E$ and $\mathfrak{g}$ is the isomorphism discussed above.

Remark. In Formula (2.6) the torsion τ_g acts only on the first factor of $T_X^{*0,1} \otimes \left(\Lambda_X^{n,q} \otimes E\right)$, since in this context the coefficients of ξ are treated as sections of $\Lambda_X^{n,q} \otimes E$, and not as E-valued (n,q)-forms. ◇

Proof of Proposition 2.2.6. Let $\psi \in \Gamma(X, \mathscr{C}^\infty(\Lambda_X^{n,q} \otimes E))$ be a section with compact support.

If $\varphi \in \Gamma(X, \mathscr{C}^\infty(\Lambda_X^{0,1} \otimes \Lambda_X^{n,q} \otimes E))$ then we have

$$\{\!\{\xi, \bar\nabla\psi\}\!\} = \int_X h_{\alpha\bar\beta} g^{I\bar J} g^{i\bar j} \xi^\alpha_{\bar j, \bar J} \left(\overline{g_{K\bar I} \frac{\partial}{\partial \bar z^i} \left(g^{K\bar L} \psi^\beta_{\bar L}\right)}\right) dV(z) = \int_X h_{\alpha\bar\beta} g^{i\bar j} \xi^\alpha_{\bar j, \bar J} \frac{\partial}{\partial z^i} \left(\overline{g^{J\bar I} \psi^\beta_{\bar I}}\right) dV(z)$$

$$= -\int_X h_{\mu\bar\beta} g^{I\bar J} h^{\mu\bar\gamma} \frac{\partial}{\partial z^i} \left(h_{\alpha\bar\gamma} g^{i\bar j} \xi^\alpha_{\bar j, \bar J}\right) \overline{\psi^\beta_{\bar I}} dV(z)$$

$$= -\int_X \left\langle \mathrm{Trace}(\mathbf{D}^{1,0}\mathfrak{g}\xi), \psi \right\rangle - \int_X h_{\alpha\bar\beta} g^{I\bar J} \frac{\partial g^{i\bar j}}{\partial z^i} \xi^\alpha_{\bar j, \bar J} \overline{\psi^\beta_{\bar I}}$$

$$= -\int_X \left\langle \mathrm{Trace}(\mathbf{D}^{1,0}\mathfrak{g}\xi), \psi \right\rangle - \int_X \left\langle \Lambda_g(\partial\omega_g)^{\sharp_g}\xi, \psi \right\rangle,$$

where the contraction of the section ξ of $\Lambda_X^{1,0} \otimes (\Lambda_X^{n,q} \otimes)$ and the $(0,1)$-vector field $\Lambda_g(\partial\omega_g)^{\sharp_g}$ is with respect to first factor $\Lambda_X^{1,0}$. The proof is complete in view of the definition of τ_g. $\square$

2.2.5 A tale of two Laplace-Beltrami operators

The $\bar\nabla$-Laplace-Beltrami operator $\Delta := \bar\nabla^*\bar\nabla$ on sections of $\Lambda_X^{n,q} \otimes E$ is given by the formula

$$\Delta\varphi = \bar\nabla^* \left(\mathfrak{g}^{-1}\bar\partial\mathfrak{g}\varphi\right) = -\mathrm{Trace}\mathbf{D}^{1,0}\bar\partial\mathfrak{g}\varphi - \tau_g\mathfrak{g}^{-1}\bar\partial\mathfrak{g}\varphi, \quad \varphi \in \Gamma(X, \mathscr{C}^\infty(\Lambda_X^{n,q} \otimes E)).$$

More important to us is the $\bar\partial$-Laplace-Beltrami operator: the formal operator defined by

$$\Box := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial.$$

We compute that

$$\bar\partial^*\bar\partial\varphi = (-1)^{n+1} g^{i\bar j} \nabla_i \left((\bar\partial\varphi)^\alpha_{\bar j \bar J} \mathbf{dz} \wedge d\bar z^J \otimes e_\alpha\right) - \tau_g\bar\partial\varphi$$

$$= (-1)^{n+1} g^{i\bar j} \nabla_i \Big((-1)^n (\nabla_{\bar j}\varphi)^\alpha_{\bar J} \mathbf{dz} \wedge d\bar z^J \otimes e_\alpha$$

$$+ (-1)^n \sum_{k=1}^q (-1)^k (\nabla_{\bar j_k}\varphi)^\alpha_{\bar j \bar j_1 \ldots \widehat{\bar j_k} \ldots \bar j_q} \mathbf{dz} \wedge d\bar z^J \otimes e_\alpha \Big) - \tau_g\bar\partial\varphi$$

$$= \left((-g^{i\bar j}\nabla_i\nabla_{\bar j}\varphi)^\alpha_{\bar j_1 \ldots \bar j_q} - \sum_{k=1}^q (g^{i\bar j}\nabla_i\nabla_{\bar j_k}\varphi)^\alpha_{\bar j_1 \ldots (\bar j)_k \ldots \bar j_q} \right) \mathbf{dz} \wedge d\bar z^J \otimes e_\alpha - (-1)^q\tau_g\bar\partial\varphi,$$

and

$$(\bar\partial\bar\partial^*\varphi)^\alpha_{\bar j_1 \ldots \bar j_q} = (-1)^n \sum_{k=1}^q (-1)^{k+1} (\nabla_{\bar j_k}(\bar\partial^*\varphi))^\alpha_{\bar j_1 \ldots \widehat{\bar j_k} \ldots \bar j_q} - (-1)^{q-1}\bar\partial\tau_g\varphi$$

$$= (-1)^n \sum_{k=1}^q (-1)^{k+1} (\nabla_{\bar j_k}((-1)^{n+1} g^{i\bar j}(\nabla_i\varphi)))^\alpha_{\bar j \bar j_1 \ldots \widehat{\bar j_k} \ldots \bar j_q} + (-1)^q\bar\partial\tau_g\varphi$$

$$= \sum_{k=1}^q (\nabla_{\bar j_k}(g^{i\bar j}\nabla_i\varphi))^\alpha_{\bar j_1 \ldots (\bar j)_k \ldots \bar j_q} + (-1)^q\bar\partial\tau_g\varphi$$

$$= \sum_{k=1}^q g^{i\bar j} (\nabla_{\bar j_k}\nabla_i\varphi)^\alpha_{\bar j_1 \ldots (\bar j)_k \ldots \bar j_q} + (-1)^q\bar\partial\tau_g\varphi,$$

Thus

$$(\Box\varphi)^{\alpha}_{\bar{J}_q} = -g^{i\bar{j}}\nabla_i\nabla_{\bar{j}}\varphi^{\alpha}_{\bar{J}_q} - \sum_{k=1}^{q} g^{i\bar{\ell}}([\nabla_i, \nabla_{\bar{j}_k}]\varphi)^{\alpha}_{\bar{j}_1\dots(\bar{\ell})_k\dots\bar{j}_q} + (-1)^q[\bar{\partial}, \tau_g]\varphi.$$

Finally, since $[\nabla_{\bar{j}}, \nabla_i] = \Theta(h)_{i\bar{j}}$ is the curvature (see Proposition 1.5.27),

$$-\sum_{k=1}^{q} g^{i\bar{\ell}}\left[\nabla_i, \nabla_{\bar{j}_k}\right]\varphi^{\alpha}_{\bar{j}_1\dots(\bar{\ell})_k\dots\bar{j}_q} = (\Theta(h)^{\sharp_g}\varphi)^{\alpha}_{\bar{j}_1\dots\bar{j}_q}.$$

Remark. The operator $(-1)^q[\bar{\partial}, \tau_g]$ is of order 0, a fact we will not use in these notes. ◇

2.2.7 Theorem (The Bochner-Kodaira-Nakano Formula).

$$\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} = \bar{\nabla}^*\bar{\nabla} + A_g + \Theta(h)^{\sharp_g} + (-1)^q[\bar{\partial}, \tau_g],$$

where A_g is a differential operator of order 1. *Moreover, if g is Kähler then $A_g = 0$ and $\tau_g = 0$, in which case*

$$\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} = \bar{\nabla}^*\bar{\nabla} + \Theta(h)^{\sharp_g}.$$

Proof. A simple computation in local coordinates shows that

$$-\mathrm{Trace}(\mathbf{D}^{1,0}\bar{\partial}\mathfrak{g}\varphi) = -g^{i\bar{j}}\nabla_i\nabla_{\bar{j}}\varphi + [g^{i\bar{j}}, \nabla_i\nabla_{\bar{j}}].$$

Since the commutator is order 1, the result follows from the computations of $\Box$ and Δ carried out above. □

Laplace operators are non-negative symmetric operators, i.e., the satisfy

$$(Av, v) \geq 0 \quad \text{and} \quad (Av, w) = (v, Aw)$$

on some dense subset. (If the second identity holds for all $v, w \in \mathrm{Domain}(A)$, we say that A is *self-adjoint*.)

In the Kähler case, Theorem 2.2.7 tells us that the $\bar{\partial}$-Laplacian $\Box$ differs from the non-negative symmetric operator Δ by an operator of order 0, namely the curvature. If one knows that the curvature is positive then it is possible to invert $\Box$. (This statement is more subtle than it might at first seem; it will be elaborated on later in the text.)

In a general Hermitian manifold the operator $\Box - \Delta$ does not always have order 0. However, Demailly discovered that there is a symmetric operator $\hat{\Delta}$ such that $\Box - \hat{\Delta}$ is an operator of order 0. In fact, $\hat{\Delta}$ is obtained from Δ by 'completing the square'.

2.3 The Hodge Theorem

2.3.1 Statement

The Hodge Theorem is the result that tells us when we can solve the Poisson equation $\Box u = \varphi$. The obstructions to solving this equation include the kernel of $\Box$, whose elements are called *harmonic forms*. However, a priori there may be other obstructions, and one of the two technical points in the proof of the Hodge Theorem is that there are no other obstructions. Another technically demanding point is that the Kernel is finite-dimensional. The third and final technical point is that the solution u smooth when φ is smooth. The precise statement of the theorem is as follows.

2.3.1 THEOREM (Hodge Theorem). *Let X be a compact complex manifold with a Hermitian metric g for its tangent bundle, and let $E \to X$ be a holomorphic vector bundle with Hermitian metric h.*

1. *The vector spaces*

$$\mathscr{H}_{p,q}(g,h) := \mathrm{Kernel}(\Box) \cap L^2_{p,q}(g,h)$$

   are finite dimensional and consist of smooth E-valued (p,q)-forms. In particular, there exists an orthogonal projection operator

$$P_{p,q} : L^2_{p,q}(g,h) \to \mathscr{H}_{p,q}(g,h).$$

2. *Given $\varphi \in L^2_{p,q}(g,h)$, there exists $\theta \in L^2_{p,q}(g,h)$ such that $\Box\theta = \varphi$ if and only if $P_{p,q}\varphi = 0$, and every such θ is smooth if φ is smooth. Moreover, there is a unique θ_o satisfying $\Box\theta_o = \varphi$ and $P_{p,q}\theta_o = 0$.*

3. *There exists a compact self-adjoint operator*

$$G_{p,q} : L^2_{p,q}(g,h) \to L^2_{p,q}(g,h),$$

   called the Green operator, such that

$$G_{p,q}\mathscr{H}_{p,q}(g,h) = 0, \quad G_{p,q}\bar{\partial} - \bar{\partial}G_{p,q-1} = G_{p,q},\bar{\partial}^* - \bar{\partial}^*G_{p,q+1} = 0,$$

   and $\mathrm{Id} : L^2_{p,q}(g,h) \to L^2_{p,q}(g,h)$ *has the so-called* Hodge Decomposition

$$\mathrm{Id} = P_{p,q} + \Box G_{p,q}.$$

2.3.2 Two direct applications

In the next two chapters we will discuss some deeper applications of the Hodge Theorem. For now, we present two corollaries that follow rather easily from the Hodge Theorem.

Harmonic representatives of cohomology classes

Consider the twisted Dolbeault cohomology groups

$$H^{p,q}_{\bar\partial}(X,E) := \frac{\text{Kernel}\left(\bar\partial : \Gamma(X,\Lambda^{p,q}(T^*_X)\otimes E) \to \Gamma(X,\Lambda^{p,q+1}(T^*_X)\otimes E)\right)}{\bar\partial\left(\Gamma(X,\Lambda^{p,q-1}(T^*_X)\otimes E)\right)}.$$

2.3.2 Theorem (Fundamental Theorem of Hodge Theory: twisted case). *Let (X,g) be a compact Hermitian manifold, let $E \to X$ a holomorphic vector bundle with Hermitian metric h, and let $\theta \in \Gamma(X,\Lambda^{p,q}_X \otimes E)$ be a smooth E-valued (p,q)-form such that $\bar\partial\theta = 0$. Then there exists a unique θ_o such that*

$$\Box\theta_o = 0 \quad \textit{and} \quad \theta - \theta_o = \bar\partial\eta$$

for some smooth $(p,q-1)$-form η. In particular, the natural inclusion

$$\mathscr{H}_{p,q}(g,h) \hookrightarrow H^{p,q}_{\bar\partial}(X,E)$$

is an isomorphism, and thus $H^{p,q}_{\bar\partial}(X,E)$ is finite dimensional.

First, slick proof. If $\eta,\zeta \in \mathscr{H}_{p,q}(g,h)$ are Dolbeault cohomologous, say $\eta - \zeta = \bar\partial\mu$, then $\bar\partial^*\bar\partial\mu = 0$ and hence $||\bar\partial\mu||^2 = (\bar\partial^*\bar\partial\mu,\mu) = 0$, so $\eta = \zeta$. Hence every Dolbeault cohomology class contains at most one harmonic form.

To prove existence, note that by the Hodge Decomposition there exist smooth forms $P_{p,q}\theta$, α and β such that

$$\theta = P_{p,q}\theta + \bar\partial\alpha + \bar\partial^*\beta.$$

Then $\bar\partial\bar\partial^*\beta = \bar\partial\theta - \bar\partial P_{p,q}\theta + \bar\partial\bar\partial\alpha = 0$, so $||\bar\partial^*\beta||^2 = (\bar\partial\bar\partial^*\beta,\beta) = 0$, and hence θ is Dolbeault cohomologous to the harmonic form $P_{p,q}\theta$. □

Second, equivalent but more instructive proof. Let $V_\theta \subset L^2_{p,q}(g,h)$ denote the closure of the affine subspace $\theta + \bar\partial(\Gamma(X,\Lambda^{p,q-1}(T^*_X)))$. Let $\theta_o \in V_\theta$ minimize the distance to the origin over all elements of V_θ. Then $\theta_o \perp \bar\partial(\Gamma(X,\Lambda^{p,q-1}(T^*_X)))$, and thus for all $\eta \in \Gamma(X,\Lambda^{p,q-1}(T^*_X))$

$$0 = (\theta_o,\bar\partial\eta) = (\bar\partial^*\theta_o,\eta).$$

Hence $\bar\partial^*\theta_o = 0$, so $\Box\theta_o = 0$. By the Hodge Theorem θ_o is smooth, so $\theta - \theta_o$ is smooth, and also equal to $\bar\partial\eta_o$ for some $\eta_o \in L^2_{p,q-1}(g,h)$. However, the smoothness of $\bar\partial\eta_o$ does not mean η_o is smooth (unless $q = 1$; see Remark 2.3.3 below).

To see that one can find a smooth η, we proceed as follows. So far we have proved that there is a sequence $\mu_j \in \Gamma(X,\Lambda^{p,q-1}(T^*_X))$ such that

$$\theta_o - \theta = \lim \bar\partial\mu_j.$$

Therefore for all $\zeta \in \mathscr{H}_{p,q}(g,h)$ one has $(\theta_o - \theta,\zeta) = \lim(\bar\partial\mu_j,\zeta) = \lim(\mu_j,\bar\partial\zeta) = 0$, which is to say, $\theta_o - \theta \perp \mathscr{H}_{p,q}(g,h)$. By the Hodge Theorem

$$\theta_o - \theta = \Box\chi = \bar\partial\bar\partial^*\chi + \bar\partial^*\bar\partial\chi$$

for some smooth $\chi \in \Gamma(X, \Lambda_X^{p,q} \otimes E)$. But

$$\bar\partial\bar\partial^*\bar\partial\chi = \bar\partial\Box\chi = \bar\partial(\theta_o - \theta) = 0,$$

and thus $0 = (\bar\partial\bar\partial^*\bar\partial\chi, \bar\partial\chi) = ||\bar\partial^*\bar\partial\chi||^2$. We see that, with $\eta := \bar\partial^*\chi$ (which is smooth),

$$\theta_o - \theta = \bar\partial\eta.$$

Finally, suppose θ_1 is another (p,q)-form such that $\Box\theta_1 = 0$ and $\theta_1 - \theta = \bar\partial\psi$ for some $(p,q-1)$-form ψ. Then $(\theta_i, \bar\partial\eta) = (\bar\partial^*\theta_i, \eta) = 0$, and thus $\theta_i \perp \bar\partial(\Gamma(X, \Lambda^{p,q}(T_X^*)))$ for $i = 0, 1$. Additionally,

$$\theta_1 - \theta_o = (\theta_1 - \theta) - (\theta_o - \theta) = \bar\partial(\psi - \eta) \in \bar\partial\Gamma(X, \Lambda^{p,q}(T_X^*)).$$

Thus

$$||\theta_o||^2 = ||\theta_1 + \theta_o - \theta_1||^2 = ||\theta_1||^2 + ||\theta_o - \theta_1||^2.$$

Interchanging 0 and 1, we see that $||\theta_1||^2 = ||\theta_o - \theta_1||^2 + ||\theta_o||^2$. Thus $||\theta_1 - \theta_o|| = 0$, as desired. □

2.3.3 Remark. The the course of the proof of Theorem 2.3.2 we claimed without proof that if $\eta \in L^2_{p,0}(g,h)$ and $\bar\partial\eta$ is smooth then η is smooth. Let us prove this claim using the Hodge Theorem. (There are also more elementary proofs.) Suppose $\bar\partial\eta$ is smooth. Then so is $\alpha = \bar\partial^*\bar\partial\eta$. On the other hand, $\bar\partial^*\eta = 0$, and thus the $\alpha = \Box\eta$. Therefore the form α

(i) is a smooth $(p,0)$-form,

(ii) lies in the image of $\Box$, and

(iii) is of the form $\alpha = \bar\partial^*\beta$ for some smooth β (namely $\beta = \bar\partial\eta$).

Since the image of $\bar\partial^*$ is orthogonal to $\mathscr{H}_{p,q}(g,h)$, the Hodge Theorem yields a smooth $(p,0)$-form $\gamma \in \Gamma(X, \Lambda^{p,0}(T_X^*))$ such that $\Box\gamma = \alpha$. Thus $\Box(\gamma - \eta) = \alpha - \Box\eta = 0$, i.e., $\gamma - \eta$ is harmonic, hence a smooth, $(p,0)$-form. (Note that a harmonic $(p,0)$-form is a holomorphic p-form, so we don't need the Hodge Theorem for this part.) Thus $\eta = \gamma + (\eta - \gamma)$ is smooth, as claimed. ◇

Kodaira-Serre Duality

The complex version of the Hodge star operator

$$\bar\star : \Gamma(X, \mathscr{C}^\infty_X(\Lambda_X^{p,q})) \to \Gamma(X, \mathscr{C}^\infty_X(\Lambda_X^{n-p,n-q}))$$

is defined by requiring that

$$\langle\theta, \eta\rangle_g \, dV_g = \theta \wedge \bar\star\eta.$$

If $\varphi^1, ..., \varphi^n$ is an orthonormal local frame for $(T^{1,0}_X)^*$ and we write $\varphi^I = \varphi^{i_1} \wedge ... \wedge \varphi^{i_p}$ and $\bar\varphi^J = \bar\varphi^{j_1} \wedge ... \wedge \bar\varphi^{j_q}$,

$$\bar\star \varphi^I \wedge \bar\varphi^J = (-1)^{q(n-p)+\frac{n^2}{2}-n} \varphi^{I^*} \wedge \bar\varphi^{J^*},$$

where $I^* \in \{1, ..., n\}^{n-p}$, $J^* \in \{1, ..., n\}^{n-q}$, $I \cup I^* = J \cup J^* = \{1, ..., n\}$ and $\mathrm{sgn}\binom{1...n}{II^*} = \mathrm{sgn}\binom{1...n}{JJ^*} = 1$. The Hodge star is then extended to $\Gamma(X, \mathscr{C}^\infty_X(\Lambda^{p,q}_X))$ by conjugate-linearity. Note that

$$\bar\star\bar\star \varphi^I \wedge \bar\varphi^J = (-1)^{n(p+q)+n^2} \varphi^I \wedge \bar\varphi^J,$$

and thus

$$\bar\star\bar\star\theta = (-1)^{n(p+q)+n^2}\theta \quad \text{for all} \quad \theta \in \Gamma(X, \mathscr{C}^\infty_X(\Lambda^{p,q}_X)).$$

Using the Hodge star operator, we find through integration by parts that

$$\begin{aligned}
\int_X \bar\partial^*\theta \wedge \bar\star\eta &:= \int_X \theta \wedge \bar\star\bar\partial\eta \\
&= \int_X \overline{\bar\partial\eta \wedge \bar\star\theta} \\
&= \int_X \partial\overline{(\eta \wedge \bar\star\theta)} + (-1)^{p+q} \int_X \overline{\eta \wedge \bar\partial\bar\star\theta} \\
&= \int_X d\overline{(\eta \wedge \bar\star\theta)} + \int_X (-\bar\star\bar\partial\bar\star)\theta \wedge \bar\star\eta = \int_X (-\bar\star\bar\partial\bar\star)\theta \wedge \bar\star\eta.
\end{aligned}$$

Thus the formal adjoint is

$$\bar\partial^* = -\bar\star\bar\partial\bar\star.$$

Similarly, one can define the twisted version of the Hodge star operator

$$\bar\star : \Gamma(X, \mathscr{C}^\infty_X(\Lambda^{p,q}_X \otimes E)) \to \Gamma(X, \mathscr{C}^\infty_X(\Lambda^{n-p,n-q}_X \otimes E^*))$$

by the identity

$$\langle \theta, \eta \rangle_{h,g} \, dV_g = \theta \wedge \bar\star\eta,$$

where h denotes the Hermitian metric for $E \to X$. The notation here requires some clarification. The wedge product on the right hand side includes the pairing between E and E^*: if we write $\theta = \theta_{I\bar J} dz^I \wedge d\bar z^J$ and $\bar\star\eta = \xi_{K\bar L} dz^K \wedge d\bar z^L$ with $\theta_{I\bar J}$ and $\xi_{K\bar L}$ local sections of E and E^* respectively then

$$\theta \wedge \bar\star\eta = \langle \theta_{I\bar J}, \xi_{K\bar L} \rangle \, dz^I \wedge d\bar z^J \wedge dz^K \wedge d\bar z^L$$

The computation of the formal adjoint $\bar\partial^*$ in the untwisted case carry through in the same way to the twisted setting. If in addition to the orthonormal frame $\varphi^1, ..., \varphi^n$ of $T^*_{X,z}$ we also fix an orthonormal frame $e_1, ..., e_r$ for E_z then

$$\bar\star \varphi^I \wedge \bar\varphi^J \otimes e_\alpha = (-1)^{q(n-p)+\frac{n^2}{2}-n} \varphi^{I^*} \wedge \bar\varphi^{J^*} \otimes e^{*\alpha},$$

where $e^{*1}, ..., e^{*r}$ denotes the dual frame for E^*_z. It follows that

$$\bar\star\bar\star\theta = (-1)^{n(p+q)+n^2}\theta.$$

Moreover, one has $\bar{\partial}^* = -\bar{\star}\bar{\partial}\bar{\star}$ for the same reason as in the untwisted case. Hence

$$\Box\bar{\star} = -\bar{\partial}\bar{\star}\bar{\partial}\bar{\star}\bar{\star} - \bar{\star}\bar{\partial}\bar{\star}\bar{\partial}\bar{\star} = -\bar{\star}\bar{\star}\bar{\partial}\bar{\star}\bar{\partial} - \bar{\star}\bar{\partial}\bar{\star}\bar{\partial}\bar{\star} = \bar{\star}\Box.$$

Thus $\bar{\star}$ maps harmonic E-valued (p,q)-forms to harmonic E^*-valued $(n-p, n-q)$-forms. We therefore obtain the following theorem.

2.3.4 Theorem (Kodaira-Serre Duality). *The operator $\bar{\star}$ yields a ring isomorphism*

$$H^{p,q}_{\bar{\partial}}(X, E) \cong H^{n-p,n-q}_{\bar{\partial}}(X, E^*).$$

Proof. It suffices to show that the pairing $H^{p,q}_{\bar{\partial}}(X, E) \otimes H^{n-p,n-q}_{\bar{\partial}}(X, E^*) \to H^{n,n}_{\bar{\partial}}(X) \cong \mathbb{C}$ given by the cup product

$$[\alpha] \smile [\beta] := \int_X \alpha \wedge \beta$$

is non-degenerate. But

$$[\alpha] \smile [\bar{\star}\alpha] = \int_X |\alpha|^2_{h,g} dV_g > 0$$

whenever $[\alpha] \neq 0$. □

2.3.3 Preparations

Some preliminary remarks about the proof

Somewhat informally speaking the Hodge Theorem 2.3.1 asserts in part that the image of the operator $\Box$ is precisely the orthogonal complement of the kernel of $\Box$. Since the operator $\Box$ is *symmetric*, i.e., it satisfies

$$(\Box\theta, \eta) = (\theta, \Box\eta),$$

the invertibility of $\Box$ on the orthogonal complement of its kernel appears to be a familiar fact of linear algebra. Of course, this 'fact' is generally true in finite-dimensional Hermitian vector spaces, which is certainly not the case in the setting of the Hodge Theorem.

Let us review the argument in finite-dimensions. Suppose $T : H \to H$ is a linear operator on a finite-dimensional Hermitian vector space. Consider the adjoint operator $T^* : H \to H$ defined by the relation

$$(T^*v, w) = (v, Tw). \tag{2.7}$$

Evidently $\text{Image}(T)^\perp = \text{Kernel}(T^*)$, i.e., we have an orthogonal decomposition

$$H = \text{Image}(T) \oplus \text{Kernel}(T^*). \tag{2.8}$$

Of course, if T is symmetric, i.e., $T^* = T$, then we get the finite-dimensional analogue of the situation above.

It seems reasonable to try to extend the finite-dimensional argument to the infinite-dimensional setting. However, one runs into several problems. First, in order to conclude

(2.8) from (2.7), one must know that the kernel of T^* and the image of T are closed subspaces. It is easy to see that Kernel(T^*) $\subset H$ is closed: if $\{v_j\} \subset$ Kernel(T^*) and $v_j \to v$ in H then

$$|T^*v|^2 = (v, TT^*v) = \lim_j (v_j, TT^*v) = \lim_j (T^*v_j, T^*v) = 0,$$

so $v \in$ Kernel(T^*). However, in the infinite-dimensional case Image(T) need not be closed.

As a possible remedy, one might try to replace H by its Hilbert space completion. If T is initially continuous then T will extend to a continuous linear map on the completion. However, in the situation under consideration $T := \Box$ is not continuous on $H := \Gamma(X, \mathscr{C}^\infty(\Lambda_X^{p,q}))$ with respect to the $L^2_{p,q}(g,h)$-topology. And then there is the finite-dimensionality of the kernel of T, which would not follow from the continuity of T.

It therefore seems somewhat miraculous that the Hodge Theorem is true.

Here is how the situation is remedied.

(o.) To simplify things, one eliminates the kernel of $\Box$ by considering in its stead the operator $\Box + \mathrm{Id}$. (This modification makes getting back to $\Box$ rather easy.)

i. The pre-Hilbert space $\Gamma(X, \mathscr{C}^\infty(\Lambda_X^{p,q} \otimes E))$ is completed to the Hilbert space $L^2_{p,q}(g,h)$.

ii. Although $\Box + \mathrm{Id} : \Gamma(X, \mathscr{C}^\infty(\Lambda_X^{p,q} \otimes E)) \to \Gamma(X, \mathscr{C}^\infty(\Lambda_X^{p,q} \otimes E))$ is not continuous, it has many extensions to operators defined on a dense subspaces of $L^2_{p,q}(g,h)$; the *domain* of the extension. For such a 'densely defined operator' F there is a procedure for defining the 'Hilbert space adjoint' F^*.[1] A simple formal computation shows that on $\Gamma(X, \mathscr{C}^\infty(\Lambda_X^{p,q} \otimes E))$ any extension F satisfies

$$F^* = (\Box + \mathrm{Id})^* = \Box + \mathrm{Id} = F,$$

but the problem is that (a) the domain of F^* might be different from the domain of F, and (b) even if the two domains are the same, it might not be the case that $F^* = F$ outside $\Gamma(X, \mathscr{C}^\infty(\Lambda_X^{p,q}))$. A densely defined operator F that is equal to its adjoint is unsurprisingly called a *self-adjoint* operator. We will show how to construct a self-adjoint extension of $\Box + \mathrm{Id}$.

iii. Now we are faced with two more problems.

iii.a. Although we can invert F on its domain (since it is bounded below by Id), we do not know that when we apply the inverse to an element of $\Gamma(X, \mathscr{C}^\infty(\Lambda_X^{p,q} \otimes E))$ we end up with an element of $\Gamma(X, \mathscr{C}^\infty(\Lambda_X^{p,q} \otimes E))$ again; a priori we only know that the inverse is in $L^2_{p,q}(g,h)$. We will show that this is indeed the case; this is the hardest part of the proof of the Hodge Theorem.

[1]The Hilbert space adjoint of an unbounded linear map $T : H_1 \to H_2$ between Hilbert spaces is defined as follows. Let $D \subset H_1$ be the domain of the operator T. We define

$$\mathrm{Domain}(T^*) := \{\eta \in H_2 \ ; \ \mathscr{L}_\eta : D \ni x \mapsto \langle Tx, \eta \rangle \text{ is bounded}\}.$$

One can extend $\mathscr{L}_\eta$ to a bounded linear functional on H_1 by defining it to be 0 on $D^\perp$. By the Riesz Representation Theorem this extension, still denoted $\mathscr{L}_\eta$, is given by $\mathscr{L}_\eta := \langle \cdot, T^*\eta \rangle$ for some unique vector $T^*\eta \in H_1$ that satisfies $||T^*\eta||_1 = ||\mathscr{L}_\eta||_{1*}$, and it is easy to see that $\eta \mapsto T^*\eta$ is linear.

iii.b. The kernel of $\Box$ corresponds to the Eigenspace of F associated to the eigenvalue 1, but nothing so indicates that this eigenspace is finite-dimensional. To obtain finite-dimensionality we show that F^{-1} is a *compact* operator. We then invoke the Spectral Theorem for Compact Self-Adjoint Operators to conclude the finite-dimensionality of the 1-eigenspace of F^{-1}, which is the 1-eigenspace of F.

Both iii.a and iii.b require the use of the theory of Sobolev spaces, which we will use without proof.

Extension of the $\bar{\partial}$-Laplacian

First extension: currents

For $\theta \in L^2_{p,q}(g,h)$ we can define the *current*, i.e., the continuous linear functional on the space $\Gamma(X, \mathscr{C}^\infty(\Lambda^{p,q}_X \otimes E))$ of (automatically compactly supported) smooth (p,q)-forms with a certain topology which we do not discuss here, by

$$(\Box\theta)(\eta) := (\theta, \Box\eta)_g.$$

One has the following estimate: if η is supported on a compact set K then

$$|\Box\theta(\eta)| \le \sup_X |\Box\eta|_{g,h} \cdot ||\theta||_{g,h} \cdot \left(\int_K dV_g\right)^{1/2} \le C_{\theta,K} \sup_K (|\eta|_{g,h} + |\nabla\eta|_{g,h} + |\nabla^2\eta|_{g,h}).$$

The latter estimate means that the linear functional $\Box\theta$ is continuous with respect to the aforementioned topology. The current $\Box\theta$ is a 'concrete' object with which we can get started.

2.3.5 Definition. If $\theta \in L^2_{p,q}(g,h)$ then the distribution $\Box\theta$ is said to be *in* $L^2_{p,q}(g,h)$ *in the sense of distributions* if there exists $\alpha \in L^2_{p,q}(g,h)$ such that

$$(\alpha, \eta) = (\theta, \Box\eta)$$

for all smooth $\eta \in L^2_{p,q}(g,h)$.

The subspace

$$\left\{\phi \in L^2_{p,q}(g,h) \ ; \ \Box\phi \in L^2_{p,q}(g,h) \text{ in the sense of distributions}\right\} \subset L^2_{p,q}(g,h)$$

of $L^2_{p,q}(X,g)$ is aptly named the *maximal domain* of $\Box$. Although we can extend $\Box$ to this larger subspace, it is not clear that the resulting operator is self adjoint. Indeed, while its Hilbert space adjoint $\Box^*$ agrees with $\Box$ on smooth compactly supported forms, it might happen that the domain of $\Box^*$ is not the same as the domain of $\Box$. There are also other properties that we need from the extension of $\Box$, that do not obviously hold if we take the maximal extension.

Instead we construct in the next paragraph a different extension.

Friedrichs Extension of $\square$

Suppose $\mathcal{H}$ is a Hilbert space with inner product $(\ ,\ )$ and Q is a Hermitian form defined on a dense subspace $\mathscr{D}$ of $\mathcal{H}$ such that $Q(x,x) \geq ||x||^2$ for all $x \in \mathscr{D}$. Suppose, furthermore, that $\mathscr{D}$ is a Hilbert space with respect to the inner product determined by Q. Let $x \in \mathcal{H}$. Then the linear functional $y \mapsto (y,x)$ on $\mathscr{D}$ is continuous with respect to Q, since

$$|(y,x)|^2 \leq ||x||^2||y||^2 \leq ||x||^2 Q(y,y).$$

Thus by the Riesz Representation Theorem there is an element $Tx \in \mathscr{D}$ such that $Q(Tx,y) = (x,y)$. The operator $T : \mathcal{H} \to \mathscr{D}$ is clearly well-defined, and the estimate

$$||Tx||^2 \leq Q(Tx,Tx) = (x,Tx) \leq ||x|| \cdot ||Tx|| \tag{2.9}$$

shows that T is bounded. Moreover, if $Tx = 0$ then $0 = Q(Tx,y) = (x,y) = 0$ for all $y \in \mathscr{D}$, so by the density of $\mathscr{D}$, $x = 0$. Finally, note that

$$(x,Ty) = Q(Tx,Ty) = \overline{Q(Ty,Tx)} = \overline{(y,Tx)} = (Tx,y),$$

so that T is also symmetric. Since T is bounded, $\text{Domain}(T^*) = \mathcal{H}$, and thus T is self-adjoint.

We let $F = T^{-1}$ and $\text{Domain}(F) = T(\mathcal{H})$. Then we check that if $x, y \in T(\mathcal{H})$ then

$$(Fx,y) = (T^{-1}x,y) = Q(x,y) = \overline{Q(y,x)} = \overline{(T^{-1}y,x)} = \overline{(Fy,x)} = (x,Fy).$$

This computation shows that $F : T(\mathcal{H}) \to \mathcal{H}$ is symmetric, but in fact, as a possibly unbounded operator $F : \mathcal{H} \to \mathcal{H}$ whose domain is $T(\mathcal{H})$, it is is self adjoint.

2.3.6 Proposition (Friedrichs). *The operator F, with its domain* $\text{Domain}(F) = T(\mathcal{H})$, *is the unique self adjoint operator with domain in $\mathscr{D}$ such that $Q(x,y) = (Fx,y)$ for all $x \in$* Domain(F) *and $y \in \mathscr{D}$. Moreover, the equation $Fu = x$ has a solution u for all $x \in \mathcal{H}$, namely, $u = Tx$.*

Proof. First note that, by construction, $u = Tx$ solves the equation $Fu = x$.

Let us prove that F is self-adjoint, i.e., that

$$T(\mathcal{H}) = \text{Domain}(F^*) := \{y \in \mathcal{H}\ ;\ T(\mathcal{H}) \ni x \mapsto (Fx,y) = Q(x,y) \text{ is bounded}\}.$$

i. If $y = Tz \in T(\mathcal{H})$ then for $x \in T(\mathcal{H})$ (and even $x \in \mathscr{D}$)

$$Q(x,y) = Q(x,Tz) = \overline{Q(Tz,x)} = \overline{(z,x)} = (x,z),$$

so $x \mapsto Q(x,y)$ is bounded on $T(\mathcal{H})$, i.e., $T(\mathcal{H}) \subset \text{Domain}(F^*)$.

ii. Conversely, suppose $y \in \text{Domain}(F^*)$, i.e., $\ell : \text{Domain}(F) \ni x \mapsto Q(x,y)$ is bounded. By extending ℓ continuously to the closure of Domain(F) in $\mathcal{H}$ and then by 0 to $\text{Domain}(F)^\perp$, we may assume ℓ is bounded on all of $\mathcal{H}$. By the Riesz Representation Theorem there exists $z \in \mathcal{H}$ such that $\ell(\zeta) = (\zeta, z)$. In particular, for $\zeta \in \mathcal{H}$ we have

$$(\zeta, Tz) = (T\zeta, z) = \ell(T\zeta) = Q(T\zeta, y) = (\zeta, y).$$

Thus $y = Tz \in T(\mathcal{H})$, which shows that $\text{Domain}(F^*) \subset T(\mathcal{H})$.

Therefore F is self-adjoint. Moreover, the above argument works for any operator F with domain in $\mathscr{D}$ and satisfying $Q(x,y) = (Fx,y)$ for all $x \in \text{Domain}(F)$ and all $y \in \mathscr{D}$, and hence F is unique. $\square$

We shall now apply Proposition 2.3.6 to construct the extension of $\square$ to a self-adjoint unbounded operator on $L^2_{p,q}(g,h)$. We take as our inner product

$$Q(\theta,\eta) = (\bar\partial\theta, \bar\partial\eta) + (\bar\partial^*\theta, \bar\partial^*\eta) + (\theta,\eta).$$

Our Hilbert space $\mathcal{H}$ will be $L^2_{p,q}(g,h)$, while $\mathscr{D}$ will be the completion of $\Gamma(X, \Lambda^{p,q}T^*_X \otimes E)$ with respect to the inner product Q. The inclusion $\iota : \Gamma(X, \Lambda^{p,q}T^*_X \otimes E) \hookrightarrow L^2_{p,q}(g,h)$ extends uniquely (by density) to a bounded linear map $\iota : \mathscr{D} \to L^2_{p,q}(g,h)$ with norm ≤ 1. In fact, $\iota : \mathscr{D} \to L^2_{p,q}(g,h)$ is injective. Indeed, Suppose $\{\theta_n\}$ is a sequence of smooth forms converging to θ in $L^2_{p,q}(g,h)$ and to 0 in $\mathscr{D}$. Then

$$||\theta||^2 = \lim ||\theta_n||^2 \le \lim Q(\theta_n, \theta_n) = 0,$$

and thus $\theta = 0$.

We are thus in a position to apply Proposition 2.3.6. Let F be the operator associated to Q. By integration by parts we find that for every pair of smooth forms θ, η,

$$Q(\theta,\eta) = ((\text{Id} + \square)\theta, \eta),$$

and thus F is a self-adjoint extension of the Hermitian operator $(\text{Id} + \square)|_{\Gamma(X,\Lambda^{p,q}T^*_X\otimes E)}$. In particular, the set of all smooth (p,q)-forms on X form a dense subset of the domain of F.

Of course, since the identity operator is regular and self-adjoint, we obtain an extension of $\square$ by setting

$$\square\theta := F\theta - \theta, \quad \theta \in \text{Domain}(F).$$

2.3.7 Remark. One can now extend the notion of Harmonic forms to include those forms in $L^2_{p,q}(g,h)$ that also lie in the kernel of (the extension of) $\square$. In principle, we might then get new, non-smooth harmonic forms. Indeed, new harmonic forms may well appear if the metric g for X is not smooth enough. However, for sufficiently smooth (e.g. $\mathscr{C}^\infty$) metric g Harmonic forms are smooth, i.e., this (or any) extension of $\square$ does not increase the size of the kernel. The smoothness of Harmonic forms is one of the main difficulties in establishing the Hodge Theorem. $\diamond$

Analytic properties of the Friedrichs extension

Gårding's Inequality and the Regularity Theorem for F

The two results proved in the present section are crucial in our proof of the Hodge Theorem.

2.3.8 Theorem (Gårding's Inequality). *Let X be a complex manifold with a Hermitian metric and let $E \to X$ be a Hermitian holomorphic vector bundle. For each compact subset $K \subset\subset X$ there is a positive constant $C = C(K)$ such that*

$$||\theta||_1^2 \le CQ(\theta,\theta).$$

for all smooth E-valued (p,q)-forms θ with support in K.

2.3.9 Corollary (Rellich Compactness). *If X is a compact complex manifold then the inclusion $\mathscr{D} \hookrightarrow L^2_{p,q}(g,h)$ is a compact linear operator.*

Proof. Gårding's inequality and the obvious inequality $Q(\theta,\theta) \lesssim ||\theta||_1^2$ for all smooth E-valued (p,q)-forms θ on X shows that $\mathscr{D}$ is quasi-isometric to the Sobolev space $W^{1,2}(X,E)_{p,q}$. By Relich's Compactness Theorem the inclusion $W^{1,2}(X,E)_{p,q} \hookrightarrow L^2_{p,q}(g,h)$ is a compact operator, and the result follows. □

Before proving Theorem 2.3.8, we perform some local calculations. Fix a relatively compact coordinate chart K in X. Let us choose a local basis $\{\omega^1,...,\omega^n\}$ of $(1,0)$-forms that is pointwise orthonormal with respect to the metric associated to g. Let $L_1,...,L_n$ be a dual basis of vector fields, i.e., $\omega^i(L_j) = \delta^i_j$. We write $\omega^I = \omega^{i_1} \wedge ... \wedge \omega^{i_p}$ and $\bar\omega^J = \bar\omega^{j_1} \wedge ... \wedge \bar\omega^{j_q}$. A (p,q)-form θ with support in K can be written $\theta = \theta_{I\bar J}\omega^I \wedge \bar\omega^J$, and the expression is unique if we assume, as we do, that the functions $\theta_{I\bar J}$ are skew-symmetric in I and J. Of course, θ is smooth if and only if the functions $\theta_{I\bar J}$ are smooth.

We can choose local coordinates at a given point p of X such that $dz^i(p) = \omega^i(p)$. In these coordinates, $L_j(p) = \frac{\partial}{\partial z^j}$, and the metric g is given by $g_{i\bar j}(z) = (\delta_{i\bar j} + O(|z-p|))dz^i \wedge d\bar z^j$. We also choose a frame for E that is orthonormal at the point p. From Propositions 2.2.2 and 2.2.5 we obtain

$$(\Box\theta)_{I\bar J} = -\left(\delta_{A\bar C}\varepsilon^{\bar\ell\bar C}_{\bar J}\varepsilon^{kA}_{N} + \delta^{M\bar P}\delta_{B\bar J}\varepsilon^{kB}_{M}\varepsilon^{\bar\ell\bar N}_{\bar P}\right) L_k\overline{L}_\ell(\theta_{I\bar N}) + ... \ .$$

In this multiple sum (over all the indices except I and J) there is a lot of cancellation. In fact, we claim that the only non-zero terms occur when $k = \ell$. Note that if the latter is the case, then in fact $J = N$. To show that there are no other terms, suppose $k \neq \ell$. Then the only possibly non-zero terms occur for $J \neq N$. But then among these, the only non-zero terms occur for $\{Jk\} = \{N\ell\}$. It follows that for some multiindex R and one of its permutations R', $J = (\ell R)$ and $N = (kR')$. But for any choice of ordering,

$$\begin{aligned}
\delta_{A\bar C}\varepsilon^{\bar\ell\bar C}_{\bar J}\varepsilon^{kA}_{N} + \delta^{M\bar P}\delta_{B\bar J}\varepsilon^{kB}_{M}\varepsilon^{\bar\ell\bar N}_{\bar P} &= \delta_{A\bar C}\varepsilon^{\bar\ell\bar C}_{\bar J}\varepsilon^{kA}_{N} + \delta^{M\bar P}\varepsilon^{kJ}_{M}\varepsilon^{\bar\ell\bar N}_{\bar P} \\
&= \delta_{A\bar C}\varepsilon^{\bar C}_{\bar R}\varepsilon^{A}_{R'} + \delta^{M\bar P}\varepsilon^{k\ell R}_{M}\varepsilon^{\bar\ell\bar k\bar R'}_{\bar P} \\
&= \delta_{A\bar C}\varepsilon^{\bar C}_{\bar R}\varepsilon^{A}_{R'} - \delta^{M\bar P}\varepsilon^{k\ell R}_{M}\varepsilon^{\bar k\bar\ell\bar R'}_{\bar P} = 0.
\end{aligned}$$

We have thus proved the following.

2.3.10 Proposition. *If θ is a smooth E-valued (p,q)-form then in any local chart, in which $\theta = \theta_{I\bar J}\omega^I \wedge \bar\omega^J$ as above, one has*

$$\Box\theta = -(L_k\overline{L_k}\theta_{I\bar J})\omega^I \wedge \bar\omega^J + \text{ lower order terms}$$

and

$$\Box\theta = -(\overline{L_k}L_k\theta_{I\bar J})\omega^I \wedge \bar\omega^J + \text{ lower order terms,}$$

where 'lower order terms' means derivatives of the functions $\theta_{I\bar J}$ of order ≤ 1.

Proof of the Gårding Inequality. Using a partition of unity, it suffices to consider a single coordinate chart in K.

Integration by parts applied to Proposition 2.3.10 then shows that

$$(\Box\theta, \theta) = \sum_{k=1}^{n} ||(L_k\theta_{IJ})\omega^I \wedge \bar{\omega}^J||^2 + O(||\theta||_1||\theta||_0)$$

and also

$$(\Box\theta, \theta) = \sum_{k=1}^{n} ||(\overline{L_k}\theta_{IJ})\omega^I \wedge \bar{\omega}^J||^2 + O(||\theta||_1||\theta||_0).$$

It follows that

$$Q(\theta, \theta) = ||\theta||_0^2 + \frac{1}{2}\left(\sum_{k=1}^{n} ||(L_k\theta_{IJ})\omega^I \wedge \bar{\omega}^J||^2 + \sum_{k=1}^{n} ||(\overline{L_k}\theta_{IJ})\omega^I \wedge \bar{\omega}^J||^2\right) + O(||\theta||_1||\theta||_0)$$

Now,

$$||\theta||_1||\theta||_0 \le \varepsilon||\theta||_1^2 + \frac{1}{4\varepsilon}||\theta||_0^2 \quad \text{and} \quad ||\theta||_0^2 \le Q(\theta, \theta).$$

Moreover, all first order derivatives are linear combinations of L_k and $\overline{L_k}$. It follows that

$$\begin{aligned} ||\theta||_1^2 &\lesssim Q(\theta, \theta) + ||\theta||_1||\theta||_0 \\ &\lesssim Q(\theta, \theta) + \varepsilon||\theta||_1^2 + \tfrac{1}{\varepsilon}||\theta||_0^2 \\ &\lesssim Q(\theta, \theta) + \varepsilon||\theta||_1^2. \end{aligned}$$

Taking $0 < \varepsilon << 1$ completes the proof. □

2.3.11 THEOREM (Regularity for F). *Let $\eta \in W^{s,2}(X, E)_{p,q}$, the Sobolev space of E-valued (p, q)-forms whose derivatives up to order s are in L^2, for some $s \ge 0$. Let $\theta = T\eta \in \mathscr{D}$ be the unique solution to the equation*

$$F\theta = \eta.$$

Then $\theta \in W^{s+2,2}(X, E)_{p,q}$. In particular, if

$$\eta \in \Gamma(X, \mathscr{C}^\infty(\Lambda_X^{p,q} \otimes E)) \quad \textit{and} \quad (I + \Box)\theta = \eta$$

in the sense of currents then $\theta \in \Gamma(X, \mathscr{C}^\infty(\Lambda_X^{p,q} \otimes E))$.

By definition of the Sobolev norm on a compact manifold the statement of Theorem 2.3.11 is purely local. Thus in order to prove Theorem 2.3.11, it is sufficient to establish the following somewhat stronger result.

2.3.12 THEOREM. *Let $U \subset X$ be an open coordinate chart, $\chi \in \mathscr{C}_0^\infty(U)$ a smooth function, and $\chi_1 \in \mathscr{C}_0^\infty(U)$ another smooth function such that $\chi_1|Support(\chi) \equiv 1$. Let $\eta \in L^2_{p,q}(g, h)$ such that $\chi_1\eta \in \Gamma(X, \mathscr{C}^\infty(\Lambda_X^{p,q} \otimes E))$, and let $\theta = T\eta \in \mathscr{D}$ be the unique solution of $F\theta = \eta$. Then there exists a constant C depending on s, χ and χ_1 but not on η such that*

$$||\chi\theta||_{s+2}^2 \le C(||\chi_1\eta||_s^2 + ||\theta||_0^2).$$

2.3.13 REMARK. Since, by definition, smooth forms in $W^{s,2}(X,E)_{p,q}$ form a dense subspace, Theorem 2.3.12 immediately implies Theorem 2.3.11. Moreover, Theorem 2.3.12 and the Sobolev Embedding Theorem imply that η is smooth wherever $F\eta$ is smooth. ⋄

We now embark on the proof of Theorem 2.3.12. Let us fix coordinate charts $V \subset\subset U \subset X$ and a $\mathscr{C}^\infty$-function χ_1 with support in U such that $\chi_1|_V \equiv 1$.

2.3.14 LEMMA. *For each $\mathscr{C}^\infty$-function χ with compact support in V,*

$$||\chi\theta||_1^2 \lesssim ||\chi_1 F\theta||^2 + ||\theta||^2$$

uniformly for $\theta \in \Gamma(U, \mathscr{C}^\infty(\Lambda_X^{p,q}))$.

Proof. Let A be a first order differential operator. Then

$$\begin{aligned}(A\chi\theta, A\chi\theta) &= (\chi A\theta, A\chi\theta) + O(||\theta||\cdot||\chi\theta||_1)\\ &= (A\theta, \chi A\chi\theta) + O(||\theta||\cdot||\chi\theta||_1)\\ &= (A\theta, A\chi^2\theta) + (A\theta, [\chi, A]\chi\theta) + O(||\theta||\cdot||\chi\theta||_1)\\ &= (A\theta, A\chi^2\theta) + (\theta, A^*[\chi, A]\chi\theta) + O(||\theta||\cdot||\chi\theta||_1)\\ &= (A\theta, A\chi^2\theta) + O(||\theta||\cdot||\chi\theta||_1).\end{aligned}$$

Here we use the fact that the formal adjoint A^* of A is a 1^{st} order differential operator, the commutator $[A,\chi]$ is a 0^{th} order operator, and there are no boundary terms because of compact support.

Taking $A = \bar\partial$ and then $A = \bar\partial^*$, we have

$$(\bar\partial\chi\theta, \bar\partial\chi\theta) = (\bar\partial\theta, \bar\partial\chi^2\theta) + O(||\theta||\cdot||\chi\theta||_1)$$

and

$$(\bar\partial^*\chi\theta, \bar\partial^*\chi\theta) = (\bar\partial^*\theta, \bar\partial^*\chi^2\theta) + O(||\theta||\cdot||\chi\theta||_1).$$

It follows that for $\varepsilon > 0$ sufficiently small there exists $C = O(1/\varepsilon)$ such that

$$\begin{aligned}Q(\chi\theta, \chi\theta) &= Q(\theta, \chi^2\theta) + O(||\theta||\cdot||\chi\theta||_1)\\ &= (F\theta, \chi\theta) + O(||\theta||\cdot||\chi\theta||_1)\\ &\leq ||\chi_1 F\theta||^2 + \varepsilon||\chi\theta||_1^2 + C||\theta||^2.\end{aligned}$$

An application of Gårding's Inequality completes the proof. □

2.3.15 REMARK. The same proof works if χ, instead of being a single function, is a matrix of smooth functions with compact support, acting on the components of θ, i.e., a smooth vector bundle endomorphism of $\Lambda_X^{p,q} \otimes E$ over U. ⋄

For the remainder of the present paragraph we use the notation $\tilde\chi$ for some function (or matrix of functions) obtained from the the function (or matrix of functions) χ; for example, any derivative of χ, including the commutator $[\bar\partial, \chi]$, etc.

2.3.16 LEMMA. *Let χ and θ be as in Lemma 2.3.14, let α be a multiindex with $|\alpha| = k$ and let D^α act on forms componentwise. Then*

$$Q(D^\alpha\chi\theta, D^\alpha\chi\theta) = Q(\theta, \chi(D^\alpha)^* D^\alpha\chi\theta) + O(||\tilde{\chi}\theta||_k||\chi\theta||_{k+1}).$$

Here $(D^\alpha)^*$ denotes the formal adjoint of D^α.

Proof. We repeatedly use the fact that the commutator of a differential operator of order k with a differential operator of order ℓ is a differential operator of order $k+\ell-1$, applied here in the case $\ell = 1$. We compute that

$$\begin{aligned}(\bar\partial D^\alpha\chi\theta, \bar\partial D^\alpha\chi\theta) &= (D^\alpha\chi\bar\partial\theta, \bar\partial D^\alpha\chi\theta) + O(||\tilde{\chi}\theta||_k||\chi\theta||_{k+1})\\ &= (\bar\partial\theta, \chi(D^\alpha)^*\bar\partial D^\alpha\chi\theta) + O(||\tilde{\chi}\theta||_k||\chi\theta||_{k+1})\\ &= (\bar\partial\theta, \bar\partial\chi(D^\alpha)^* D^\alpha\chi\theta) + (\bar\partial\theta, [\chi(D^\alpha)^*, \bar\partial]D^\alpha\chi\theta) + O(||\tilde{\chi}\theta||_k||\chi\theta||_{k+1})\\ &= (\bar\partial\theta, \bar\partial\chi(D^\alpha)^* D^\alpha\chi\theta) + ([\bar\partial^*, \chi D^\alpha,]\bar\partial\theta, D^\alpha\chi\theta) + O(||\tilde{\chi}\theta||_k||\chi\theta||_{k+1})\\ &= (\bar\partial\theta, \bar\partial\chi(D^\alpha)^* D^\alpha\chi\theta) + O(||\tilde{\chi}\theta||_k||\chi\theta||_{k+1}).\end{aligned}$$

Repeating the calculation with $\bar\partial$ replaced by $\bar\partial^*$ yields

$$(\bar\partial^* D^\alpha\chi\theta, \bar\partial^* D^\alpha\chi\theta) = (\bar\partial^*\theta, \bar\partial^*\chi(D^\alpha)^* D^\alpha\chi\theta) + O(||\tilde{\chi}\theta||_k||\chi\theta||_{k+1}).$$

Adding these and observing that $(\theta, \chi(D^\alpha)^* D^\alpha\chi\theta) = ||D^\alpha\chi\theta||^2$ completes the proof. □

Proof of Theorem 2.3.12. For $s = 0$ we use Lemma 2.3.14 to get

$$||\chi\theta||_2^2 \sim \sum_j ||\partial_{x^j}\chi\theta||_1^2 + ||\chi\theta||_1^2 \lesssim \sum_j ||\partial_{x^j}\chi\theta||_1^2 + ||\chi_1 F\theta||^2 + ||\theta||^2.$$

But by Gårding's Inequality and Lemma 2.3.16,

$$\begin{aligned}||\partial_{x^j}\chi\theta||_1^2 &\lesssim Q(\partial_{x^j}\chi\theta, \partial_{x^j}\chi\theta)\\ &= Q(\theta, \chi(\partial_{x^j})^*\partial_{x^j}\chi\theta) + O(||\tilde{\chi}\theta||_1||\chi\theta||_2)\\ &= (F\theta, \chi(\partial_{x^j})^*\partial_{x^j}\chi\theta) + O(||\tilde{\chi}\theta||_1||\chi\theta||_2)\\ &= (\chi_1 F\theta, \chi(\partial_{x^j})^*\partial_{x^j}\chi\theta) + O(||\tilde{\chi}\theta||_1||\chi\theta||_2)\\ &\lesssim ||\chi_1 F\theta||^2 + ||\tilde{\chi}\theta||_1^2 + \varepsilon||\chi\theta||_2^2\\ &\lesssim ||\chi_1 F\theta||^2 + \varepsilon||\chi\theta||_2^2.\end{aligned}$$

Taking ε sufficiently small gives us the case $s = 0$.

By induction, suppose the result has been proved for $s - 1$. Then

$$||\chi\theta||_{s+2}^2 \sim \sum_j ||D^\beta\chi\theta||_1^2 + ||\chi\theta||_{s+1}^2 \lesssim \sum_j ||D^\beta\chi\theta||_1^2 + ||\tilde{\chi}F\theta||_{s-1}^2 + ||\theta||^2.$$

But now

$$\begin{aligned}
||D^\alpha \chi\theta||_1^2 &\lesssim Q(D^\alpha\chi\theta, D^\alpha\chi\theta)\\
&= Q(\theta, \chi(D^\alpha)^* D^\alpha\chi\theta) + O(||\tilde{\chi}\theta||_{s+1}||\chi\theta||_{s+2})\\
&= (F\theta, \chi(D^\alpha)^* D^\alpha\chi\theta) + O(||\tilde{\chi}\theta||_{s+1}||\chi\theta||_{s+2})\\
&= (\chi_1 F\theta, \chi(D^\alpha)^* D^\alpha\chi\theta) + O(||\tilde{\chi}\theta||_{s+1}||\chi\theta||_{s+2})\\
&\leq ||\chi_1 F\theta||_s||\chi(D^\alpha)^* D^\alpha\chi\theta||_{-s} + O(||\tilde{\chi}\theta||_{s+1}||\chi\theta||_{s+2})\\
&\lesssim ||\chi_1 F\theta||_s^2 + ||\tilde{\chi}\theta||_{s+1} + \varepsilon||\chi\theta||_{s+2}^2\\
&\lesssim ||\chi_1 F\theta||_s^2 + ||\chi_1 F\theta||_{s-1} + \varepsilon||\chi\theta||_{s+2}^2\\
&\lesssim ||\chi_1 F\theta||_s^2 + \varepsilon||\chi\theta||_{s+2}^2.
\end{aligned}$$

In the second-to-last estimate we used the inductive hypothesis. Putting everything together yields

$$||\chi\theta||_{s+2}^2 \lesssim ||\tilde{\chi}F\theta||_s^2 + ||\theta||^2 + \varepsilon||\chi\theta||_{s+2}^2.$$

Taking ε sufficiently small completes the proof. □

2.3.4 Proof of the Hodge Theorem

Smoothness of Harmonic forms

Fix $\lambda \in \mathbb{R}$. Suppose $\theta \in W^{s,2}(X,E)_{p,q}$ satisfies $\Box\theta = \lambda\theta$. Then by definition, $F\theta = (\lambda+1)\theta$. It follows from Lemma 2.3.12 that $\theta \in W^{s+2,2}(X,E)_{p,q}$, and thus by induction

$$\theta \in \bigcap_{s\geq 0} W^{s,2}(X,E)_{p,q}.$$

By the Sobolev Embedding Theorem $\theta \in \Gamma(X, \Lambda^{p,q}(T_X^*) \otimes E)$. In particular, taking $\lambda = 0$ proves the smoothness part of Item 1 of the Hodge Theorem 2.3.1. Finite dimensionality of $\mathscr{H}_{p,q}(g,h)$ (and, in fact, of all the eigenspaces of F) will be handled below.

Solutions are orthogonal to harmonic forms

We turn next to the necessity of the condition in Item 2 of Theorem 2.3.1. Let $\varphi \in L^2_{p,q}(g,h)$ and suppose we want to solve the equation $\Box u = \varphi$. If a solution u exists then one has

$$(\varphi, \theta) = (\Box u, \theta) = (u, \Box\theta).$$

In particular, $\varphi \perp \text{Kernel}(\Box)$.

Compactness of the the inverse of F

Recall the operator $T = F^{-1} : L^2_{p,q}(g,h) \to \mathscr{D}$ defined in Section 2.3.3. As we showed on page 76, T is bounded. Since the composition of a bounded operator and a compact operator is a compact operator, Corollary 2.3.9 implies the following lemma.

2.3.17 LEMMA. *Let $\iota : \mathscr{D} \hookrightarrow L^2_{p,q}(g,h)$ denote the inclusion. Then the operator*

$$L := \iota \circ T : L^2_{p,q}(g,h) \to L^2_{p,q}(g,h)$$

is compact.

END OF THE PROOF OF THE HODGE THEOREM

Since $||\cdot||_{L^2} \le ||\cdot||_{\mathscr{D}}$, the estimate (2.9) implies that $||T\xi||_{L^2} \le ||\xi||_{L^2}$. Thus the eigenvalues of L are no more than 1. Moreover, since F is positive-definite and self-adjoint, so is L, and thus all the eigenvalues of L are positive.

By the Spectral Theorem for compact self-adjoint operators there is a discrete set of numbers $1 = \lambda_1 > \lambda_2 > ....$ such that $\lambda_m \to 0$ and the subspaces

$$\mathscr{K}_m(X,E) := \{\theta \in \Gamma(X, \Lambda^{p,q}_X \otimes E) \; ; \; L\theta = \lambda_m \theta\}$$

are finite dimensional and provide a decomposition of $L^2_{p,q}(g,h)$, i.e.,

$$L^2_{p,q}(g,h) = \bigoplus_{m \ge 1} \mathscr{K}_m(X,E)$$

in the sense of Hilbert spaces (meaning that every element on the left hand side is a possibly infinite sum of elements of the components on the right, convergent with respect to the Hilbert norm). Observe that for any $\theta \in \mathscr{K}_m(X)$,

$$\Box\theta = (\lambda_m^{-1} - 1)\theta = \frac{1-\lambda_m}{\lambda_m}\theta.$$

In particular, $\mathscr{K}_1(X,E) = \mathscr{H}_{p,q}(g,h)$.

Let us fix bases

$$\left\{\theta_j^{(m)} \; ; \; 1 \le j \le N_m\right\} \subset \mathscr{K}_m(X,E), \quad m = 1, 2, ... \, .$$

Let $\varphi \in L^2_{p,q}(g,h)$. Decompose φ orthogonally in $L^2_{p,q}(g,h)$ as

$$\varphi = \sum_{m=1}^{\infty} \sum_{j=1}^{N_m} c_{jm}\theta_j^{(m)}$$

Let

$$G_{p,q}\varphi := \sum_{m=2}^{\infty} \sum_{j=1}^{N_m} \frac{\lambda_m c_{jm}}{1-\lambda_m}\theta_j^{(m)}.$$

Then $G_{p,q}\mathscr{H}_{p,q}(g,h) = \{0\}$. Moreover, the inequality

$$\frac{|\lambda_m|}{|1-\lambda_m|} \le \frac{1}{1-\lambda_2}, \quad m \ge 2$$

show that $G_{p,q}\varphi$ is convergent and

$$||G_{p,q}\varphi||_{L^2} \leq \frac{1}{1-\lambda_2}||\varphi||_{L^2}.$$

Next,

$$\Box G_{p,q}(\varphi - P_{p,q}\varphi) = \sum_{m=2}^{\infty}\sum_{j=1}^{N_m}\frac{\lambda_m c_{jm}}{1-\lambda_m}\Box\theta_j^{(m)} = \sum_{m=2}^{\infty}\sum_{j=1}^{N_m} c_{jm}\theta_j^{(m)} = \varphi - P_{p,q}\varphi.$$

Since $\bar\partial\Box = \Box\bar\partial$ and $\bar\partial^*\Box = \Box\bar\partial^*$, the operators $\bar\partial$ and $\bar\partial^*$ map eigenforms of a given eigenvalue to eigenforms of the same eigenvalue. It follows that $G_{p,q}\bar\partial = \bar\partial G_{p,q-1}$ and $G_{p,q}\bar\partial^* = \bar\partial^* G_{p,q+1}$.

It remains to show that $G_{p,q}$ is compact. But the formula for $G_{p,q}$, coupled with the fact that $\lambda_m \to 0$, shows that for any $\varepsilon > 0$ there is a finite-rank (therefore compact) operator A_ε and an operator B_ε whose norm is at most ε, such that

$$G_{p,q} = A_\varepsilon + B_\varepsilon. \tag{2.10}$$

Indeed, we choose m_o such that $\lambda_{m_o} < \varepsilon(1-\lambda_{m_o})$ and take

$$A_\varepsilon \sum_{m=2}^{m_o}\sum_{j=1}^{N_m} c_{jm}\theta_j^{(m)} := \sum_{m=2}^{m_o}\sum_{j=1}^{N_m}\frac{\lambda_m c_{jm}}{1-\lambda_m}\theta_j^{(m)}.$$

But any operator is compact if and only if it has a decomposition of the form (2.10). Hence $G_{p,q}$ is compact, and the Hodge Theorem is proved. $\Box$

2.3.5 Laplacians of Kähler manifolds and the $\partial\bar\partial$-Lemma

The crucial feature of Kähler manifolds that we will use below is the characterization that a metric g on X is Kähler if and only if each point of X lie is a holomorphic coordinate system with respect to which the metric is Euclidean to second order (Theorem 1.5.21).

2.3.18 Lemma. *On a Kähler manifold the operators $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ and $\Delta_\partial = \partial\partial^* + \partial^*\partial$ agree, and are equal to $\frac{1}{2}\Delta$, where $\Delta = dd^* + d^*d$ is the real Laplace Beltrami Operator. In particular, $\partial\Box = \Box\partial$.*

Proof. We restrict to (p,q)-forms. In the untwisted case the Bochner-Kodaira Identity reads

$$\Box = \bar\nabla^*\bar\nabla + \Theta(g^{(p,0)} \otimes \det g)^{\sharp_g}.$$

A straightforward repetition of the proof of the Bochner-Kodaira Identity for the operator ∂ in place of $\bar\partial$ (which only makes sense in the untwisted case) shows that

$$\Delta_\partial = \nabla^*\nabla + \Theta(g^{(0,q)} \otimes \det g)^{\sharp_g}.$$

But then

$$\Box - \Delta_\partial = -g^{i\bar{j}}(\nabla_i \nabla_{\bar{j}} - \nabla_{\bar{j}} \nabla_i) + \Theta(g^{(\cdot)} \otimes \det g)^{\bar{\sharp}_g} - \Theta(g^{(\cdot)} \otimes \det g)^{\sharp_g},$$

and the right hand side vanishes by direct computation. Therefore $\Box = \Delta_\partial$.

Next, since $d = \partial + \bar{\partial}$,

$$\Delta = \Box + \Delta_\partial + \bar{\partial}\partial^* + \partial^*\bar{\partial} + \partial\bar{\partial}^* + \bar{\partial}^*\partial = \Box + \Delta_\partial + \overline{\partial\bar{\partial}^* + \bar{\partial}^*\partial} + \partial\bar{\partial}^* + \bar{\partial}^*\partial.$$

We wish to show that $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$. Since (as the reader can verify) $\partial\bar{\partial}^* + \bar{\partial}^*\partial$ is a first order differential operator, we may study it in Kähler coordinates, i.e., coordinates in which the metric is Euclidean to second order. Using Proposition 2.2.5 (note that the covariant derivatives are now the usual partial derivatives, since we are in the untwisted case) and the formula

$$\partial\varphi = \varepsilon_M^{(kI)} \frac{\partial \varphi_{I\bar{J}}}{\partial z^k} dz^M \wedge d\bar{z}^J,$$

we have

$$(\partial\bar{\partial}^*\varphi)_{M\bar{J}} = (-1)^p \varepsilon_M^{(kI)} \delta^{\ell\bar{j}} \frac{\partial}{\partial z^k} \frac{\partial}{\partial z^\ell} \varphi_{I\bar{j}\bar{J}} = -(-1)^{p+1} \delta^{\ell\bar{j}} \frac{\partial}{\partial z^\ell} \varepsilon_M^{(kI)} \frac{\partial}{\partial z^k} \varphi_{I\bar{j}\bar{J}} = -(\bar{\partial}^*\partial\varphi)_{M\bar{J}}.$$

Finally, if $\Box = \Delta_\partial$ then $\partial\Box = \partial\Delta_\partial = \Delta\partial = \Box\partial$. The proof is complete. □

With the aid of the Hodge Theorem, we can now use the previous lemma to prove the following important result.

2.3.19 THEOREM ($\partial\bar{\partial}$-Lemma). *Let X be a compact Kähler manifold and let φ be a d-closed (p,q)-form on X. Assume moreover that φ is d-, ∂- or $\bar{\partial}$-exact. Then there exists a $(p-1,q-1)$-form ψ on X such that $\partial\bar{\partial}\psi = \varphi$. Moreover, if $p = q$ and φ is Hermitian then the exists such a ψ so that $\sqrt{\text{-}1}\psi$ is Hermitian.*

Proof. Since $\Box = \Delta_\partial = \frac{1}{2}\Delta$, the kernels of these operators are all the same, and consist of harmonic forms.

Let us assume first that φ is $\bar{\partial}$-exact. Then φ is orthogonal to the harmonic forms, and by the Hodge Theorem

$$\varphi = (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})G_{\bar{\partial}}(\varphi) = \bar{\partial}\bar{\partial}^* G_{\bar{\partial}}(\varphi),$$

where $G_{\bar{\partial}}$ is the Green operator of $\Box$. The last equality holds because $\bar{\partial}$ and $G_{\bar{\partial}}$ commute.

Now, since $0 = d\varphi = \partial\varphi + \bar{\partial}\varphi$, we must have $\partial\varphi = 0$ by type. Thus the form $\eta = \bar{\partial}^* G_{\bar{\partial}}(\varphi)$ satisfies

$$\partial\eta = -\bar{\partial}^*\partial G_{\bar{\partial}}(\varphi) = -\bar{\partial}^* G_{\bar{\partial}}(\partial\varphi) = 0.$$

Since $\eta \in \text{Image}(\bar{\partial}^*)$, η is orthogonal to the harmonic forms, and thus

$$\eta = (\partial\partial^* + \partial^*\partial)G_\partial(\eta) = \partial\partial^* G_\partial(\eta),$$

where again the last equality holds because ∂ and G_∂ commute. Thus we have

$$\varphi = \bar{\partial}(\bar{\partial}^* G_{\bar{\partial}}(\varphi)) = \bar{\partial}\eta = \bar{\partial}\partial(\partial^* G_\partial(\eta)) = \bar{\partial}\partial(\partial^*\bar{\partial}^* G_\partial G_{\bar{\partial}}(\varphi)),$$

which establishes the case of $\bar{\partial}$-exact φ. The case of ∂-exact is the same, with ∂ and $\bar{\partial}$ interchanged in the argument. And since any d-exact form is a sum of ∂-exact and $\bar{\partial}$-exact forms, the d-exact case follows as well.

Finally, if φ is Hermitian then

$$\sqrt{\text{-}1}\partial^*\bar{\partial}^* G_{\partial} G_{\bar{\partial}}(\varphi)$$

is Hermitian. The proof is complete. □

An important application of the $\partial\bar{\partial}$-Lemma is the following result.

2.3.20 THEOREM. *Let X be a compact Kähler manifold and $L \to X$ a holomorphic line bundle. Let ω be a Hermitian $(1,1)$-form in the cohomology class $c_1(L)$. Then there is a metric $e^{-\varphi}$ for L such*

$$\frac{\sqrt{\text{-}1}}{2\pi}\partial\bar{\partial}\varphi = \omega.$$

Proof. Fix any smooth metric $e^{-\varphi_o}$ for L. Then $\theta := \omega - \frac{\sqrt{\text{-}1}}{2\pi}\partial\bar{\partial}\varphi_o$ is Hermitian and d-cohomologically trivial. It follows from the $\partial\bar{\partial}$-Lemma 2.3.19 that

$$\theta = \frac{\sqrt{\text{-}1}}{2\pi}\partial\bar{\partial} f$$

for some real function f, and then the metric $e^{-(\varphi_o + f)}$ has curvature ω. □

2.4 $\mathbf{L^2}$ Estimates for the $\bar{\partial}$ Equation

In complex analytic geometry one often tries to construct holomorphic functions, or holomorphic sections of a given vector bundle. The existence of such functions already implies that the vector bundle in question (which is the trivial line bundle if we are considering functions) admits a possibly singular Hermitian metric with some sort of positivity.

By contrast, if a holomorphic vector bundle $E \to X$ has a Hermitian metric with sufficiently positive curvature, one would like to construct sections of this holomorphic vector bundle. The usual technique is to first construct a smooth section, and then correct this section to be holomorphic.

One can check whether a smooth section s is holomorphic by examining $\bar{\partial}s$; s is holomorphic if and only if $\bar{\partial}s = 0$. If a linear correction $s - u$ of s is holomorphic then u must be a solution of the equation

$$\bar{\partial}u = \bar{\partial}s.$$

The right hand side is a given E-valued $(0,1)$-form, and one wants to choose u so that the desired properties of s are not destroyed after subtracting u.

We shall use this philosophy in the next two chapters. In the present chapter, we focus on a specific way of solving the equation $\bar{\partial}u = f$ for a given $\bar{\partial}$-closed E-valued $(0,1)$-form (or more generally, (p,q)-form). There are several approaches to the problem, but the most general seems to be obtaining a solution with L^2 estimates, and this is the approach we shall pursue.

2.4.1 Compact Kähler Manifolds

In the case of a compact manifold, all smooth forms have compact support. Thus the methods of the 1-dimensional case can be applied to the setting of compact manifolds, resulting in a proof of Hörmander's Theorem that, as the reader will later see, is much less technical than the proof of Hörmander's Theorem in the case of general pseudoconvex Kähler manifolds.

Though there is a definite need for the general case of Hörmander's Theorem in some of the deeper applications to algebraic and complex geometry, the compact case already has applications to important results in algebraic geometry and in the geometry of compact Kähler manifolds.

Hörmander's Theorem on compact Kähler manifolds

2.4.1 Theorem. *Let (X, g) be a compact Kähler manifold and let $E \to X$ be a holomorphic vector bundle with Hermitian metric h. Fix p, q with $0 \le p \le n$ and $1 \le q \le n$. Assume that, as linear transformations acting on E-valued (p, q)-forms pointwise,*

$$\Theta(g^{(p)} \otimes \det(g) \otimes h)^{\bar{\sharp}_g} \ge c \cdot \mathrm{Id} \tag{2.11}$$

for some positive constant c. Then for every $\bar\partial$-closed E-valued (p, q)-form φ such that

$$\int_X |\varphi|^2_{g,h} dV_g < +\infty$$

there exists a E-valued $(p, q-1)$-form u such that

$$\bar\partial u = \varphi \quad \textit{and} \quad \int_X |u|^2_{g,h} dV_g \le \frac{1}{c} \int_X |\varphi|^2_{g,h} dV_g.$$

Moreover, $H^{p,q}_{\bar\partial}(X, E) = \{0\}$.

Remark. Note that the vanishing of $H^{p,q}_{\bar\partial}(X, E)$ does not follow immediately from the solvability of $\bar\partial$, because we are only producing square-integrable solutions. In fact, even for smooth data φ there are always non-smooth solutions u of the equation $\bar\partial u = \varphi$ when $q > 1$.

Although we could use the Functional Analysis Lemma 2.1.6, we will give a direct proof that uses similar ideas from functional analysis, but that will produce for us a smooth solution when the data is smooth. ◇

Proof. Let φ be the form given in the hypotheses. For any smooth, E-valued (p, q)-form ψ The Bochner-Kodaira-Nakano Identity implies the estimate

$$||\psi||^2 \le \frac{1}{c} \left(||\bar\partial \psi||^2 + ||\bar\partial^* \psi||^2 \right). \tag{2.12}$$

Now consider the bilinear form $(\cdot, \cdot)_{\mathscr{H}}$ defined on smooth sections of $\Lambda^{p,q}_X \otimes E$ by

$$(\psi_1, \psi_2)_{\mathscr{H}} := (\bar\partial \psi_1, \bar\partial \psi_2) + (\bar\partial^* \psi_1, \bar\partial^* \psi_2).$$

The inequality (2.12) implies that $||\cdot||_{\mathscr{H}}$ is a norm. Define $\mathscr{H}$ to be the Hilbert space completion of $\Gamma(X, \mathscr{C}^\infty(\Lambda_X^{p,q} \otimes E))$ with respect to the norm $||\cdot||_{\mathscr{H}}$.

Let $\lambda_\varphi : L^2_{p,q}(g,h) \to \mathbb{C}$ by

$$\lambda_\varphi(\psi) := (\psi, \varphi).$$

By Cauchy-Schwarz

$$|\lambda_\varphi(\psi)|^2 \le \frac{||\varphi||^2}{c} ||\psi||^2_{\mathscr{H}},$$

and thus $\lambda_\varphi \in \mathscr{H}^*$. Therefore by the Riesz Representation Theorem there exists $v \in \mathscr{H}$ such that

$$||v||^2_{\mathscr{H}} \le \frac{||\varphi||^2}{c} \quad \text{and} \quad (v,\psi)_{\mathscr{H}} = \lambda_\varphi(\psi), \quad \psi \in \mathscr{H}.$$

The latter means that

$$(\bar\partial v, \bar\partial \psi) + (\bar\partial^* v, \bar\partial^* \psi) = (\varphi, \psi), \quad \psi \in \mathscr{H}. \tag{2.13}$$

In particular, the latter holds for smooth ψ. Thus $\Box v = \varphi$ in the weak sense. We also note that, by Theorem 2.3.12, v lies in the Sobolev space $W^{1,2}$.

Now, since $\bar\partial\varphi = 0$ we claim that $\bar\partial v = 0$. Indeed, for any smooth form θ one has

$$0 = (\varphi, \bar\partial^*\theta) = (\bar\partial v, \bar\partial\bar\partial^*\theta) + (\bar\partial^* v, \bar\partial\bar\partial^*\theta) = (\bar\partial v, \bar\partial\bar\partial^*\theta) = (\bar\partial^*\bar\partial v, \bar\partial^*\theta).$$

Choosing a sequence of smooth forms v_j that converges to v in $W^{1,2}$ and setting $\theta := \bar\partial v_j$, we have

$$0 = \lim_j (\bar\partial^*\bar\partial v, \bar\partial^*\bar\partial v_j) = ||\bar\partial^*\bar\partial v||^2,$$

which proves that $\bar\partial^*\bar\partial v = 0$, and hence $||\bar\partial v||^2 = (\bar\partial^*\bar\partial v, v) = 0$, which proves our claim.

With the vanishing of $\bar\partial v$ in hand, (2.13) shows that the twisted form $u := \bar\partial^* v$ satisfies

$$(u, \bar\partial^*\psi) = (\varphi, \psi), \quad \text{i.e., } \bar\partial u = \varphi \quad \text{(in the weak sense)}$$

and

$$||u||^2 = ||v||^2_{\mathscr{H}} \le \frac{||\varphi||^2}{c}.$$

Finally, suppose φ is smooth. Then by the Hodge Theorem 2.3.1.2 the solution v of $\Box v = \varphi$ is smooth, and thus so is $u = \bar\partial^* v$. In particular, by definition of $H^{p,q}_{\bar\partial}(X,E)$, $H^{p,q}(X,E) = \{0\}$. □

2.4.2 Remark. The vanishing of the cohomology in Theorem 2.4.1 can also be deduced from the Hodge Theorem and the formal Bochner-Kodaira Identity. Indeed, by Theorem 2.3.2 every cohomology class in $H^{n,q}_{\bar\partial}(X,E)$ contains a unique harmonic form. If φ is such a harmonic form then by the Bochner-Kodaira Identity we have

$$0 = \Box\varphi = \bar\nabla^*\bar\nabla\varphi + \Theta(h)^{\sharp_g}\varphi.$$

Taking inner product with φ yields

$$0 = (\Box\varphi, \varphi) = ||\bar\nabla\varphi||^2 + (\Theta(h)^{\sharp_g}\varphi, \varphi) \ge c||\varphi||^2,$$

and thus $\varphi = 0$ (and also $\bar\nabla\varphi = 0$, i.e., harmonic forms are ‘parallel’). ◇

If we unravel the right hand side of the assumption (2.11) in Theorem 2.4.1 we discover the formula

$$\begin{aligned}\left(\Theta(g^{(p,0)}\otimes \det g\otimes h)^{\bar{\sharp}_g}\varphi\right)^{\alpha}_{i_1\dots i_p\bar{j}_1\dots\bar{j}_q} := &-\sum_{k=1}^{q} g^{i\bar{\ell}}\Theta^{\alpha}_{\beta i\bar{j}_k}\varphi^{\beta}_{I_p\bar{j}_1\dots(\bar{\ell})_k\dots\bar{j}_q}\\ &+\sum_{k=1}^{q}\sum_{\nu=1}^{p} R_{i_\nu}{}^{r\bar{\ell}}{}_{\bar{j}_k}\varphi^{\alpha}_{i_1\dots(r)_\nu\dots i_p\bar{j}_1\dots(\bar{\ell})_k\dots\bar{j}_q}\\ &-\sum_{k=1}^{q} R^{\bar{s}}{}_{\bar{j}_k}\varphi^{\alpha}_{I_p\bar{j}_1\dots(\bar{s})_k\dots\bar{j}_q}.\end{aligned}$$

It is hard to tell from this expression whether the assumption (2.11) holds for given metrics g and h. A few special cases are easier to handle.

Kodaira Vanishing

Let $L\to X$ be a holomorphic line bundle on a complex manifold X of complex dimension n, and let $e^{-\varphi}$ be a smooth Hermitian metric for L. The positivity of the curvature $\partial\bar{\partial}\varphi$ of $e^{-\varphi}$ means precisely that the $(1,1)$-form

$$\omega := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$$

is a Kähler form. (Thus, *a posteriori*, every complex manifold admitting a line bundle with a metric of positive curvature is necessarily Kähler.) We therefore can, and do, use the Kähler form ω to define our Kähler metric g on X.

Suppose β is a smooth, L-valued (n,q)-form. Then

$$\begin{aligned}&\Theta(e^{-\varphi})^{\bar{\sharp}_g}\beta\\ &= 2\pi\sum_{k=1}^{q} g^{i\bar{\ell}}g_{i\bar{j}_k}\beta_{\bar{j}_1\dots(\bar{\ell})_k\dots\bar{j}_q}dz^1\wedge\dots\wedge dz^n\wedge d\bar{z}^{j_1}\wedge\dots\wedge d\bar{z}^{j_q}\\ &= 2\pi q\beta\end{aligned}$$

By Theorem 2.4.1 we have the following theorem.

2.4.3 Theorem (Kodaira Vanishing Theorem). *Let X be a compact Kähler manifold and let $L\to X$ be a holomorphic line bundle admitting a smooth metric of positive curvature. Then*

$$H^{n,q}_{\bar{\partial}}(X,L)=\{0\}.$$

Vanishing theorems for $q>1$

Let (X,ω) be a Kähler manifold. Suppose $L\to X$ is a line bundle with smooth Hermitian metric $e^{-\varphi}$ whose curvature $\Omega=\sqrt{-1}\partial\bar{\partial}\varphi$ is not necessarily positive. Let

$$\lambda_1(p)\le\lambda_2(p)\le\dots\le\lambda_n(p)$$

denote the eigenvalues of θ_p relative to ω_p, thought of as Hermitian forms on $T^{1,0}_{X,p}$. (These eigenvalues depend on the point p, but nature of this dependence will not be important to us.) Fix a correspoinding ω-orthonormal basis $\xi_1(p), ..., \xi_n(p)$ of eigenvectors for $T^{1,0}_{X,p}$, and let $\alpha^i \in T^{1,0*}_{X,p}$ be defined by

$$\alpha^i(v) = \omega(\xi_i, v), \quad v \in T^{1,0}_{X,p}.$$

Given an L-valued (n,q)-form β, we can write $\beta = \beta_{\bar J}\alpha^1 \wedge ... \wedge \alpha^n \wedge \bar\alpha^J \otimes e$. We compute that

$$\begin{aligned}\Theta(e^{-\varphi})^{\sharp_g}\beta &= \sum_{k=1}^{q} g^{i\bar\ell}\Omega_{i\bar j_k}\beta_{\bar j_1 ...(\bar\ell)_k ... \bar j_q}\alpha^1 \wedge ... \wedge \alpha^n \wedge \bar\alpha^J \otimes e \\ &= \sum_{k=1}^{q} \delta^{i\bar\ell}\lambda_i\delta_{i\bar j_k}\beta_{\bar j_1 ...(\bar\ell)_k ... \bar j_q}\alpha^1 \wedge ... \wedge \alpha^n \wedge \bar\alpha^J \otimes e \\ &= (\lambda_{j_1} + ... + \lambda_{j_q})\beta_{\bar J}\alpha^1 \wedge ... \wedge \alpha^n \wedge \bar\alpha^J \otimes e.\end{aligned}$$

Since $\omega = \sqrt{\text{-}1}\sum_i \alpha^i \wedge \bar\alpha^i$ and $\Omega = \sqrt{\text{-}1}\sum_i \lambda_i \alpha^i \wedge \bar\alpha^i$, we have

$$\Omega \wedge \omega^{q-1} = (\sqrt{\text{-}1})^q (q-1)! \sum_{|J|=q} (\lambda_{j_1} + \cdots + \lambda_{j_q})\alpha^{j_1} \wedge \bar\alpha^{j_1} \wedge \cdots \wedge \alpha^{j_q} \wedge \bar\alpha^{j_q}$$

and

$$\omega^q = (\sqrt{\text{-}1})^q q! \sum_{|J|=q} \alpha^{j_1} \wedge \bar\alpha^{j_1} \wedge \cdots \wedge \alpha^{j_q} \wedge \bar\alpha^{j_q}.$$

By Theorem 2.4.1 we have the following theorem.

2.4.4 Theorem. *Let (X, ω) be a compact Kähler manifold, let $L \to X$ be a holomorphic line bundle with smooth metric $e^{-\varphi}$, and let $q \in \{1, ..., n\}$. If there exists a constant $c > 0$ such that*

$$\sqrt{\text{-}1}\partial\bar\partial\varphi \wedge \omega^{q-1} \ge c\omega^q.$$

Then

$$H^{n,q}_{\bar\partial}(X, L) = \{0\}.$$

2.4.5 Remark. Note that if $q = 1$ then $\sqrt{\text{-}1}\partial\bar\partial\varphi$ satisfies the hypotheses of Theorem 2.4.4 if and only if $\sqrt{\text{-}1}\partial\bar\partial\varphi$ is positive definite, and the latter condition is independent of ω. On the other hand, if $q > 1$ the hypotheses of Theorem 2.4.4 depend both on $\sqrt{\text{-}1}\partial\bar\partial\varphi$ and ω. ⋄

Positivity for vector bundles and Nakano Vanishing

Continuing in the case $p = n$ on a Kähler manifold (X, g), let us pass to the case of a higher rank vector bundle E with Hermitian metric h.

Recalling that the curvature $\Theta(h)$ of the Chern connection for h is a linear map from $\Gamma(X, E)$ to $\Gamma(X, \Lambda^{1,1}T^*_X \otimes E)$, we can define a quadratic form on the fibers of $T^{1,0}_X \otimes E \to X$ by

$$\langle \xi \otimes v, \eta \otimes w \rangle_{h,\Theta(h)} := h(\Theta(h)_{\xi,\bar\eta}v, w).$$

2.4.6 Definition. Let $E \to X$ be a holomorphic vector bundle of rank r with Hermitian metric h on a complex manifold X of complex dimension n. Fix a Hermitian metric g on X.

(i) We say that h has positive curvature in the sense of Griffiths at a point $x \in X$ if there exists $c > 0$ such that

$$\langle \xi \otimes v, \xi \otimes v \rangle_{h,\Theta(h)} \geq c \cdot h(v,v) g(\xi, \xi)$$

for all $v \otimes \xi \in T^{1,0}_{X,x} \otimes E_x$.

(ii) We say that h has positive curvature in the sense of Nakano if there exists $c > 0$ such that

$$\left\langle \sum_{j=1}^{\rho} \xi_j \otimes v_j, \sum_{k=1}^{\rho} \xi_k \otimes v_k \right\rangle_{h,\Omega(h)} \geq c \sum_{j,k=1}^{\rho} h(v_j, v_j) g(\xi_k, \xi_k)$$

for all $\xi_1 \otimes v_1, ..., \xi_\rho \otimes v_\rho \in T^{1,0}_{X,x} \otimes E_x$, where $\rho = \min(n, r)$.

2.4.7 Remark. Elements of fibers $(T^{1,0}_X \otimes E)_x = T^{1,0}_{X,x} \otimes E_x$ can be seen as linear maps from $T^{1,0}_{X,x}$ to E^*_x. The rank of each such linear map is called the *rank* of the tensor. The rank of a non-zero tensor can be any integer between 1 and $\min(\dim_{\mathbb{C}}(X), \mathrm{rank}(E))$. A tensor has rank 1 if and only if it is *indecomposable*, i.e., of the form $\xi \otimes v$ for some $\xi \in T^{1,0}_{X,x}$ and some $v \in E_x$.

The different notions of positivity can be explained more clearly in terms of rank. The metric $\mathfrak{h}$ for $E \to X$ is Griffiths positive (resp. non-negative) if and only if the quadratic form $\{\cdot, \cdot\}$ is positive (resp. non-negative) on all tensor of rank 1. Nakano positivity for $\mathfrak{h}$ is equivalent to the positivity of $\{\cdot, \cdot\}$.

Demailly has introduced intermediate notions of positivity. One says that $\mathfrak{h}$ has k-positive curvature in the sense of Demailly if $\{\cdot, \cdot\}$ is positive on all tensor of rank k. ◇

Clearly these notions of positivity do not depend on the choice of the metric g. Moreover, we have the following theorem.

2.4.8 Theorem (Nakano Vanishing Theorem). *Let X be a compact Kähler manifold and let $E \to X$ be a holomorphic vector bundle admitting a smooth metric whose curvature is positive in the sense of Nakano. Then*

$$H^{n,q}_{\bar{\partial}}(X, E) = \{0\}$$

for all $q \geq 1$.

Akizuki-Nakano Vanishing

Let $E \to X$ be a holomorphic vector bundle over the Kähler manifold (X, g). For general (p, q), the Bochner-Kodaira Identity reads

$$\begin{aligned}(\Box\varphi)^{\alpha}_{i_1\dots i_p\bar{j}_1\dots\bar{j}_q} &= -g^{i\bar{j}}\nabla_i\nabla_{\bar{j}}\varphi^{\alpha}_{i_1\dots i_p\bar{j}_1\dots\bar{j}_q} + \sum_{k=1}^{q} g^{i\bar{\ell}}\Omega^{\alpha}_{\beta i\bar{j}_k}\varphi^{\beta}_{I_p\bar{j}_1\dots(\bar{\ell})_k\dots\bar{j}_q} \\ &+ \sum_{k=1}^{q}\sum_{\nu=1}^{p} R_{i_\nu}{}^{r\bar{\ell}}{}_{\bar{j}_k}\varphi^{\alpha}_{i_1\dots(r)_\nu\dots i_p\bar{j}_1\dots(\bar{\ell})_k\dots\bar{j}_q} \\ &+ \sum_{k=1}^{q} R^{\bar{s}}{}_{\bar{j}_k}\varphi^{\alpha}_{I_p\bar{j}_1\dots(\bar{s})_k\dots\bar{j}_q}.\end{aligned}$$

for any smooth E-valued (p, q)-form φ.

When the vector bundle $E \to X$ is trivial, we also have a well-defined ∂ operator, but in general we must use its covariant version, which we have denoted $\nabla^{1,0}$. Let us derive a formula for the covariant ∂-Laplacian

$$\Delta^{1,0} := \Delta_{\nabla^{1,0}} = \nabla^{1,0}(\nabla^{1,0})^* + (\nabla^{1,0})^*\nabla^{1,0}$$

for (p, q)-forms with values in a holomorphic vector bundle $E \to X$. Note that the derivation is not completely symmetric to the derivation of a formula for $\Box$, since the vector bundle $E \to X$ is holomorphic, and not anti-holomorphic.

We can carry out the derivation in the same way as for $\Box$, so we will be sketchy, leaving verification of fine details to the reader. Like Propositions 2.2.2 and 2.2.5, we have

$$(\nabla^{1,0}\varphi)^{\alpha}_{i_o i_1\dots i_p\bar{J}} = \sum_{k=0}^{p}(-1)^k\nabla_{i_k}\varphi^{\alpha}_{i_o\dots\hat{i_k}\dots i_p}$$

and

$$((\nabla^{1,0})^*\varphi)^{\alpha}_{i_1\dots i_{p-1}\bar{J}} = -g^{i\bar{j}}\nabla_{\bar{j}}\varphi^{\alpha}_{ii_1\dots i_{p-1}\bar{J}}.$$

Therefore we have

$$((\nabla^{1,0})^*\nabla^{1,0}\varphi)^{\alpha}_{i_1\dots i_p\bar{J}} = -g^{i\bar{j}}\nabla_{\bar{j}}\nabla_i\varphi^{\alpha}_{i_1\dots i_p\bar{J}} + \sum_{k=1}^{p} g^{i\bar{j}}\nabla_{\bar{j}}\nabla_{i_k}\varphi^{\alpha}_{i_1\dots(i)_k\dots i_p\bar{J}}$$

and

$$(\nabla^{1,0}(\nabla^{1,0})^*\varphi)^{\alpha}_{i_1\dots i_p\bar{J}} = -\sum_{k=1}^{p} g^{i\bar{j}}\nabla_{i_k}\nabla_{\bar{j}}\varphi^{\alpha}_{i_1\dots(i)_k\dots i_p\bar{J}},$$

and thus

$$(\Delta^{1,0}\varphi)^{\alpha}_{i_1\dots i_p\bar{J}} = -g^{i\bar{j}}\nabla_{\bar{j}}\nabla_i\varphi^{\alpha}_{i_1\dots i_p\bar{J}} - \sum_{k=1}^{p} g^{i\bar{j}}[\nabla_{i_k}, \nabla_{\bar{j}}]\varphi^{\alpha}_{i_1\dots(i)_k\dots i_p\bar{J}}.$$

We can then compute, along the same lines as in the proof of Theorem **??**, that

$$(\Delta^{1,0}\varphi)^\alpha_{i_1\dots i_p\bar J} = -g^{i\bar j}\nabla_{\bar j}\nabla_i\varphi^\alpha_{i_1\dots i_p\bar J} - \sum_{k=1}^q g^{i\bar\ell}\Omega^\alpha_{\beta i\bar j_k}\varphi^\beta_{i_1\dots(\ell)_k\dots i_p\bar J}$$
$$+\sum_{k=1}^q\sum_{\nu=1}^p R_{i_\nu}{}^{r\bar\ell}{}_{\bar j_k}\varphi^\alpha_{i_1\dots(r)_\nu\dots i_p\bar j_1\dots(\bar\ell)_k\dots\bar j_q}$$
$$+\sum_{k=1}^q R^{\bar s}{}_{\bar j_k}\varphi^\alpha_{I_p\bar j_1\dots(\bar s)_k\dots\bar j_q}.$$

Since

$$(g^{i\bar j}[\nabla_i,\nabla_{\bar j}]\varphi)^\alpha_{I\bar J} = g^{i\bar j}\Omega^\alpha_{\beta i\bar j}\varphi^\beta_{I\bar J}$$

is the trace of the curvature in the base directions, we finally conclude that

$$(\Box\varphi - \Delta^{1,0}\varphi)^\alpha_{I\bar J} = \sum_{k=1}^p g^{\ell\bar j}\Omega^\alpha_{\beta i_k\bar j}\varphi^\beta_{i_1\dots(\ell)_k\dots i_p\bar J} + \sum_{k=1}^q g^{i\bar\ell}\Omega^\alpha_{\beta i\bar j_k}\varphi^\beta_{I\bar j_1\dots(\bar\ell)_k\dots\bar j_q} - g^{i\bar j}\Omega^\alpha_{\beta i\bar j}\varphi^\beta_{I\bar J}.$$

In particular, if E is a line bundle, we have the following result.

2.4.9 Proposition. *Let X be a Kähler manifold and let $L\to X$ be a holomorphic line bundle with Hermitian metric $e^{-\varphi}$. Then with $\Omega = \frac{\partial^2\varphi}{\partial z^i\partial\bar z^j}dz^i\wedge d\bar z^j$,*

$$\Box\varphi - \Delta^{1,0}\varphi = \sum_{k=1}^p g^{\nu\bar j}\Omega_{i_k\bar j}\varphi_{i_1\dots(\nu)_k\dots i_p\bar J} + \sum_{k=1}^q g^{i\bar\ell}\Omega_{i\bar j_k}\varphi_{I_p\bar j_1\dots(\bar\ell)_k\dots\bar j_q} - g^{i\bar j}\Omega_{i\bar j}\varphi.$$

2.4.10 Remark. If we take L to be the trivial line bundle with the trivial metric $e^{-\varphi}=1$ then we recover the identity

$$\Box\varphi = \Delta_\partial\varphi$$

discussed in the previous section. ◇

We can finally prove the following generalization of the Kodaira Vanishing Theorem.

2.4.11 Theorem (Akizuki-Nakano Vanishing Theorem). *Let X be a compact Kähler manifold and let $L\to X$ be a holomorphic line bundle admitting a smooth metric of positive curvature. Then*

$$H^{p,q}_{\bar\partial}(X,L) = \{0\}$$

for all $p+q>n$.

Proof. As in the proof of Theorem 2.4.3, we take a positively curved metric $e^{-\varphi}$ for $L\to X$ and let g be the metric whose metric form is $\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi$. Then by Proposition 2.4.9 the identity

$$\Box\varphi = \Delta^{1,0}\varphi + (p+q-n)\varphi$$

holds for all smooth L-valued (p,q)-forms. It follows that

$$||\bar\partial\varphi||^2 + ||\bar\partial^*\varphi||^2 = ||\nabla^{1,0}\varphi||^2 + ||(\nabla^{1,0})^*\varphi||^2 + (p+q-n)||\varphi||^2 \geq (p+q-n)||\varphi||^2$$

for all smooth L-valued (p,q)-forms. If the latter estimate replaces (2.11) in the statement of Theorem 2.4.1 then, with $c=p+q-n>0$, the proof of Theorem 2.4.1 goes through. □

2.4.2 Complete Kähler manifolds

Let (X, g) be a Hermitian manifold and let $E \to X$ be a holomorphic vector bundle with smooth Hermitian metric h. We define the Hilbert spaces $L^2_{p,q}(g, h)$ to be the Hilbert space completion of the space $\Gamma_o(X, \Lambda^{p,q}_X \otimes E)$ of smooth compactly supported E-valued (p, q)-forms. (This time, since X is not smooth, we must restrict to compactly supported forms.)

The maximal extension of $\bar\partial$

We have the operator $\bar\partial : \Gamma(X, \Lambda^{p,q} T^*_X \otimes E) \to \Gamma(X, \Lambda^{p,q+1} T^*_X \otimes E)$ defined on smooth sections, and we extend $\bar\partial$ to a densely-defined operator $L^2_{p,q}(g, h) \to L^2_{p,q+1}(g, h)$ following the same path, which we now recall.

Since every element of $L^2_{p,q}(g, h)$ is trivially locally integrable, we can extend $\bar\partial$ as on operator on all of $L^2_{p,q}(g, h)$ in the sense of currents: for $\varphi \in L^2_{p,q}(g, h)$,

$$\bar\partial\varphi(\psi) := (\varphi, \bar\partial^*\psi), \quad \psi \in \Gamma_o(X, \Lambda^{p,q+1} T^*_X \otimes E),$$

where $\bar\partial^*$ is the formal adjoint of $\bar\partial$.

We can construct a densely defined extension of $\bar\partial$ if we take the domain of the extension of $\bar\partial$ to be any subspace H of $L^2_{p,q}(g, h)$ containing $\Gamma_o(X, \Lambda^{p,q} T^*_X \otimes E)$ such that for each $\varphi \in H$ the E-valued $(p, q + 1)$-current $\bar\partial\varphi$ is represented by integration against some form $F_\varphi \in L^2_{p,q+1}(g, h)$, meaning

$$\bar\partial\varphi(\psi) = (F_\varphi, \psi) \quad \text{for all } \psi \in \Gamma_o(X, \Lambda^{p,q+1}_X \otimes E).$$

(We will shorten the terminology and simply say that $\bar\partial\varphi \in L^2_{p,q+1}(g, h)$ in the sense of currents, or simply that $\bar\partial\varphi \in L^2_{p,q+1}(g, h)$.) Each such subspace H yields a different densely defined operator.

In this chapter, we will take the so-called *maximal extension* $T_{p,q}$ of $\bar\partial$ defined by the domain

$$\text{Domain}(T_{p,q}) := \{\varphi \in L^2_{p,q} \ ; \ \bar\partial\varphi \in L^2_{p,q+1}(g, h)\}.$$

2.4.12 Proposition. *The operator $T = T_{p,q}$ is closed.*

Proof. Let $\{\varphi_j\} \subset \text{Domain}(T)$ be a sequence that converges to some $\varphi \in \text{Domain}(T)$ such that $T\varphi_j \to \Phi$ in $L^2_{p,q+1}(g, h)$. Then for all $\eta \in \Gamma_o(X, \Lambda^{p,q+1}_X \otimes E)$ we have

$$(T\varphi - \Phi, \eta) = \lim(T\varphi - T\varphi_j, \eta) = \lim(\varphi - \varphi_j, \bar\partial^*\eta) = 0.$$

Thus $T\varphi = \Phi$. □

The Hilbert space adjoint $T^*_{p,q}$

By Section 2.1, the maximal extension $T_{p,q}$ of $\bar\partial$ has a Hilbert space adjoint. We recall that

$$\text{Domain}(T^*_{p,q}) = \{\alpha \in L^2_{p,q+1}(g, h) \ ; \ |(\alpha, T_{p,q}\psi)| \le C_\alpha ||\psi|| \text{ for all } \psi \in \text{Domain}(T_{p,q})\}.$$

Below we will need the following simple proposition.

2.4.13 PROPOSITION. *If* $\chi \in \mathscr{C}_o^\infty(X)$ *and* $\varphi \in \text{Domain}(T^*_{p,q}) \cap \text{Domain}(T_{p,q+1})$ *then*

$$\chi\varphi \in \text{Domain}(T^*_{p,q}) \cap \text{Domain}(T_{p,q+1}).$$

Moreover,

$$T_{p,q+1}(\chi\varphi) = \chi T_{p,q+1}\varphi + \bar{\partial}\chi \wedge \varphi \quad \textit{and} \quad T^*_{p,q}(\chi\varphi) = \chi T^*\varphi - \text{grad}''\chi \lrcorner \varphi,$$

where, for a function f, $\text{grad}''f$ *is the* $(0,1)$*-vector field defined by*

$$g(\xi, \text{grad}''f) = \partial f(\xi), \qquad \xi \in T^{1,0}_X.$$

Proof. We compute that, as currents,

$$\bar{\partial}(\chi\varphi) = \bar{\partial}\chi \wedge \varphi + \chi T_{p,q+1}\varphi,$$

so clearly $\chi\varphi \in \text{Domain}(T_{p,q+1})$ and the formula for $T_{p,q+1}$ holds. Next, if $\psi \in \text{Domain}(T_{p,q})$ then the calculation just completed shows that $\chi\psi \in \text{Domain}(T_{p,q})$. Thus

$$\begin{aligned} |(\chi\varphi, T_{p,q}\psi)| &= |(\varphi, T_{p,q}(\bar{\chi}\psi) - \bar{\partial}\bar{\chi} \wedge \psi)| \\ &\le C_\varphi ||\bar{\chi}\psi|| + ||\varphi|| \cdot ||\bar{\partial}\bar{\chi} \wedge \psi|| \\ &\le (C_\varphi \sup|\chi| + ||\varphi|| \sup_X |\partial\chi|)||\psi||, \end{aligned}$$

which shows that $\chi\varphi \in \text{Domain}(T^*_{p,q})$ and

$$(\chi\varphi, T_{p,q}\psi) = (\varphi, T_{p,q}(\bar{\chi}\psi)) - (\varphi, \bar{\partial}\bar{\chi} \wedge \psi) = (\chi T^*_{p,q}\varphi, \psi) - (\text{grad}''\chi \lrcorner \varphi, \psi),$$

and the formula for $T^*_{p,q}$ follows. □

COMPLETE KÄHLER METRICS AND THE BASIC ESTIMATE

Our next goal is to extend Hörmander's Theorem from compact Kähler manifolds to more general complete Kähler manifolds. The intuitive idea is that when X has a complete Kähler metric its boundary is infinitely far away, so partitions of unity can be brought to bear when the forms being studied are square-integrable.

Complete Riemannian manifolds and exhaustion functions

Recall that in Riemannian geometry one has the notion of a complete (connected) Riemannian manifold (M, g): The Riemannian metric g induces a distance function, with the distance $\delta_g(x, y)$ between two points $x, y \in M$ being the infimum of the lengths of any two paths connecting those two points. (In general, a curve realizing this infimum, which is called the minimizing geodesic, need not exist.) The underlying manifold with this distance function is a metric space, and we say that a Riemannian manifold is complete if this induced metric space is complete.

The celebrated Hopf-Rinow Theorem says that the completeness property of the Riemannian manifold is equivalent to the condition that for some (and hence any) $x_o \in M$ the function

$$\psi_o : M \ni x \mapsto \delta_g(x, x_o) \in [0, \infty)$$

is proper. In general the function ψ_o is not smooth, but it is Lipschitz with constant 1, and thus it is almost everywhere differentiable. Moreover, $|d\psi_o|_g \le 1$. We can therefore smooth ψ_o to a function ψ satisfying

$$|d\psi|_g \le 2 \quad \text{and} \quad |\psi(x) - \psi_o(x)| \le 1.$$

The function $\psi : M \to [-1, \infty)$ is then also proper.

It is not hard to show, on the other hand, that the Riemannian manifold (M, g) is complete if there exists a smooth proper function $\psi : M \to [A, \infty)$ on M such that $|d\psi|_g$ is uniformly bounded.

2.4.14 Remark. Note that the constant A plays no role in the completeness of (M, g), but we must have it if we insist on using the word 'proper'. In complex analysis, it is customary to avoid this trivial issue by introducing the notion of an exhaustion function. ◇

We summarize the discussion in the following proposition.

2.4.15 Proposition. *A Riemannian manifold (M, g) is complete if and only if there exists an exhaustion function $\psi \in \mathscr{C}^\infty(M)$ such that $|d\psi|_g \le 1$.*

Approximation of twisted (p,q)-forms on complete Kähler manifolds

For the rest of the chapter, we employ the following convention: we fix p and q, and let

$$T := T_{p,q-1} \quad \text{and} \quad S := T_{p,q}.$$

Since the operator of primary interest is T, while the operator S comes in to take care of the compatibility condition for the data, it is useful to distinguish these two operators more clearly, as this notation does.

On a complete Kähler manifold forms in Domain$(S) \cap$ Domain(T^*) can be approximated by smooth forms, with respect to the norm

$$\varphi \mapsto ||\varphi|| + ||T^*\varphi|| + ||S\varphi||. \tag{2.14}$$

2.4.16 Theorem. *Let (X, ω) be a complete Kähler manifold and let $E \to X$ be a holomorphic Hermitian vector bundle. Then for any E-valued (p,q)-form $\varphi \in$ Domain$(T^*) \cap$ Domain(S) there exist smooth, compactly supported E-valued (p,q)-forms $\{\varphi_k\}$ such that*

$$\lim_{k\to\infty} ||\varphi - \varphi_k|| + ||T^*\varphi - \bar\partial^*\varphi_k|| + ||S\varphi - \bar\partial\varphi_k|| = 0.$$

Proof. Fix an exhaustion function $\psi \in \mathscr{C}^\infty(X)$ such that $|d\psi|_\omega \le 1$ and a function $\chi \in \mathscr{C}^\infty(\mathbb{R})$ such that $\chi(r) = 1$ for $r \le 0$ and $\chi(r) = 0$ for $r \ge 1$. Let $f_k(x) := \chi(\psi(x) - k + 1)$. Then $f_k \in \mathscr{C}^\infty_o(X)$, and in fact f_k is supported in the compact set $X_k := \{x \in X\ ;\ \psi(x) \le k\}$. For any $\varphi \in \text{Domain}(T^*) \cap \text{Domain}(S)$ Proposition 2.4.13 implies that the compactly supported form

$$\Phi_k := f_k \varphi$$

lies in $\text{Domain}(S) \cap \text{Domain}(T^*)$ and satisfies

$$S\Phi_k = f_k S\varphi + \bar\partial f_k \wedge \varphi \quad \text{and} \quad T^*\Phi_k = f_k T^*\varphi - (\text{grad}^{0,1} f_k) \lrcorner \varphi.$$

We estimate that

$$||S\Phi_k - S\varphi|| \le ||(1 - f_k)S\varphi|| + (\sup_X |df_k|)||\mathbf{1}_{X - X_k}\varphi|| \le C(||\mathbf{1}_{X-X_k}\varphi|| + ||\mathbf{1}_{X-X_k} S\varphi||),$$

and the right hand side converges to 0 as $k \to \infty$ because φ and $S\varphi$ are in L^2. Since

$$||\text{grad}^{0,1} f_k|| = ||\partial f_k|| = \sqrt{2}||df_k||,$$

a similar calculation shows that $||T^*\Phi_k - T^*\varphi|| \to 0$ as $k \to \infty$. Therefore

$$||\varphi - \Phi_k|| + ||T^*\varphi - T^*\Phi_k|| + ||S\varphi - S\Phi_k|| \to 0 \quad \text{as } k \to \infty.$$

Using a partition of unity $\{\chi_i\}$, we can write $\Phi_k = \Phi_k^1 + \cdots + \Phi_k^N$ with each $\Phi_k^i := \chi_i f_k \varphi$ supported in some coordinate chart U^i, and, again by Proposition 2.4.13, lying in the domain of T^*. Using mollifiers in the coordinate chart U_i we can approximate Φ_k^i by a compactly supported smooth form φ_k^i that converges to Φ_k^i in the Sobolev 1-norm. Thus with $\varphi_k := \varphi_k^1 + \cdots + \varphi_k^N$ we have

$$||\Phi_k - \varphi_k|| + ||T^*\Phi_k - \bar\partial^*\varphi_k|| + ||S\Phi_k - \bar\partial\varphi_k|| \lesssim \sum_{i=1}^N ||\Phi_k^i - \varphi_k|| + ||d(\Phi_k^i - \varphi_k)|| \to 0$$

as $k \to \infty$. Thus by the triangle inequality

$$||\varphi - \varphi_k|| + ||T^*\varphi - \bar\partial^*\varphi_k|| + ||S\varphi - \bar\partial\varphi_k|| \to 0$$

as $k \to \infty$. The proof is complete. □

2.4.17 Remark. The norm (2.14) is often called the *graph norm*, since it is the restriction of the norm of $L^2_{p,q} \oplus L^2_{p,q-1} \oplus L^2_{p,q+1}$ to the graph of the operator $T^* \oplus S$. ◇

The basic estimate for complete Kähler metrics

We can now prove the following version of the Basic Estimate.

2.4.18 THEOREM (The Basic Estimate: Complete Kähler metric case). *Let (X, g) be a complete Kähler manifold and let $E \to X$ be a holomorphic vector bundle with Hermitian metric h. Then for all E-valued (p,q)-forms $\varphi \in$ Domain$(T^*) \cap$ Domain(S) one has the estimate*

$$||T^*\varphi||^2 + ||S\varphi||^2 \geq (\Theta(g^{(p,0)} \otimes \det g \otimes h)^{\bar{\sharp}_g}\varphi, \varphi).$$

Proof. By Theorem 2.4.16 the smooth compactly supported E-valued forms are dense, and thus the result follows from integration by parts applied to the Bochner-Kodaira-Nakano Identity and the obvious inequality $||\bar{\nabla}\varphi||^2 \geq 0$. □

HÖRMANDER'S THEOREM FOR COMPLETE KÄHLER METRICS

We shall now use Theorem 2.4.18 to establish the following result.

2.4.19 THEOREM (Hörmander's Theorem; complete metric case). *Let (X, g) be a complete Kähler manifold and let $E \to X$ be a holomorphic vector bundle with Hermitian metric h. Assume that*

$$\Theta(g^{(p)} \otimes \det g \otimes h)^{\sharp_g} \geq c\mathrm{Id}_{\Lambda_X^{p,q} \otimes E}$$

for some $c > 0$. Then for each E-valued (p,q)-form φ such that

$$\bar{\partial}\varphi = 0 \quad \textit{and} \quad \int_X |\varphi|^2_{h,g}\omega^n < +\infty$$

there exists a E-valued $(p, q-1)$-form u such that

$$\bar{\partial}u = \varphi \quad \textit{and} \int_X |u|^2_{h,g}\omega^n \leq \frac{1}{c}\int_X |\varphi|^2_{h,g}\omega^n.$$

Proof. Since Kernel(S) is a closed subspace of $L^2_{p,q}(g,h)$ we can replace the latter by the former. Restricting the Basic Estimate to Kernel(S) then gives us that for all $\psi \in$ Domain$(T^*) \cap$ Kernel(S),

$$|(\varphi, \psi)|^2 \leq ||\varphi||^2||\psi||^2 \leq \frac{||\varphi||^2}{c}||T^*\varphi||^2, \tag{2.15}$$

where the last inequality follows from Theorem 2.4.18. The Functional Analysis Lemma 2.1.6 with $H_1 = L^2_{p,q}(g,h)$ and $H_2 =$ Kernel(S) then yields a (p,q)-form u such that $Tu = \varphi$ and $||u|| \leq c^{-1}||\varphi||$, as desired. □

COMPARISON OF L^2 NORMS FROM COMPARISON OF METRICS

Let X be a complex manifold and $E \to X$ a holomorphic vector bundle with Hermitian metric h. Fix Hermitian metrics g and γ with metric forms ω and θ respectively, and assume $\gamma \geq g$.

Choose an ω-orthonormal basis of $(1,0)$-forms $\alpha^1, ..., \alpha^n$ such that

$$\theta = \lambda_1\sqrt{-1}\alpha^1 \wedge \bar{\alpha}^1 + \cdots + \lambda_n\sqrt{-1}\alpha^n \wedge \bar{\alpha}^n.$$

Then $\lambda_i \geq 1$ for $1 \leq i \leq n$.

Given a E-valued (p, q)-form η, one can locally write

$$\eta = \eta_{I\bar{J}} \alpha^I \wedge \bar{\alpha}^J$$

with $\eta_{I\bar{J}}$ local sections of E. With the notation $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_p}$ and $\lambda_{\bar{J}} = \lambda_{j_1} \cdots \lambda_{j_q}$, the norms of η with respect to the metrics (g, h) and (γ, h) are respectively

$$|\eta|^2_{h,g} = \frac{1}{p!q!} \sum_{|I|=p,|J|=q} |\eta_{I\bar{J}}|^2_h \quad \text{and} \quad |\eta|^2_{h,\gamma} = \frac{1}{p!q!} \sum_{|I|=p,|J|=q} \frac{|\eta_{I\bar{J}}|^2_h}{\lambda_I \lambda_{\bar{J}}}.$$

In particular,

$$|\eta|^2_{h,\gamma} \leq |\eta|^2_{h,g}.$$

On the other hand, the volume forms for the two metrics are

$$dV_g = (\sqrt{\text{-}1})^n \alpha^1 \wedge \bar{\alpha}^1 \wedge \cdots \alpha^n \wedge \bar{\alpha}^n \quad \text{and} \quad dV_\gamma = \lambda_1 \cdots \lambda_n (\sqrt{\text{-}1})^n \alpha^1 \wedge \bar{\alpha}^1 \wedge \cdots \alpha^n \wedge \bar{\alpha}^n,$$

so $dV_g \leq dV_\gamma$, and hence it is in general hard to compare the L^2-norms of (p, q)-forms with respect to these two metrics.

However, there is one exceptional but important case: the case $p = n$. In this case

$$\begin{aligned} |\eta|^2_{h,\gamma} dV_\gamma &= \frac{1}{n!q!} \sum_{|I|=n,|J|=q} \frac{|\eta_{I\bar{J}}|^2_h}{\lambda_{\bar{J}}} (\sqrt{\text{-}1})^n \alpha^1 \wedge \bar{\alpha}^1 \wedge \cdots \alpha^n \wedge \bar{\alpha}^n \\ &\leq \frac{1}{n!q!} \sum_{|I|=n,|J|=q} |\eta_{I\bar{J}}|^2_h (\sqrt{\text{-}1})^n \alpha^1 \wedge \bar{\alpha}^1 \wedge \cdots \alpha^n \wedge \bar{\alpha}^n \\ &= |\eta|^2_{h,g} dV_g. \end{aligned}$$

(The case $q = n$ also works. However, a moment's thought shows that we must view q as fixed, but that we can manipulate p because it can be pushed into the holomorphic vector bundle E.) Consequently one has the comparison of L^2-norms: if η is a E-valued (n, q)-form and $\gamma \geq g$ then

$$||\eta||^2_{h,\gamma} \leq ||\eta||^2_{h,g}.$$

This monotonicity of norms is very useful if one wants to generalize Hörmander's Theorem 2.4.19 to the setting in which the manifold X is complete Kähler but the metric g is not necessarily complete. Indeed, of g is a Hermitian (resp. Kähler) metric and g_* is a complete Hermitian (resp. Kähler) metric then for every $\varepsilon > 0$ the metric $g_\varepsilon := g + \varepsilon g_*$ is a complete Hermitian (resp. Kähler) metric that dominates g.

There is also of course the problem that the monotonicity of L^2-norms only works when $p = n$, but this matter is dealt with by another clever trick.

Hörmander's Theorem for complete Kähler manifolds

We can now remove the hypothesis of completeness for the metric g in Theorem 2.4.19.

2.4.20 Theorem (Hörmander's Theorem). *Let X be a complete Kähler manifold with metric g that is not necessarily complete, and let $E \to X$ be a holomorphic vector bundle with Hermitian metric h. Assume that*

$$\Theta(g^{(p,0)} \otimes \det g \otimes h)^{\sharp_g} \geq c\mathrm{Id}_{\Lambda^{p,q}_X \otimes E} \tag{2.16}$$

for some $c > 0$. Then for each E-valued (p, q)-form φ such that

$$\int_X |\varphi|^2_{h,g}\omega^n < +\infty \quad \textit{and} \quad \bar{\partial}\varphi = 0$$

in the sense of distributions, there exists a E-valued $(p, q-1)$-form u such that

$$\bar{\partial}u = \varphi \quad \textit{and} \int_X |u|^2_{h,g}\omega^n \leq \frac{1}{c}\int_X |\varphi|^2_{h,g}\omega^n.$$

Proof. Fix a complete Kähler metric g_* and write $g_\varepsilon := g + \varepsilon g_*$. If $\varepsilon > 0$ is sufficiently small then there exist constants $c_\varepsilon > 0$ such that

$$\lim_{\varepsilon\to 0} c_\varepsilon = c \quad \text{and} \quad \Theta(g^{(p,0)} \otimes \det g \otimes h)^{\sharp_{g_\varepsilon}} \geq c_\varepsilon \mathrm{Id}_{\Lambda^{p,q}_X \otimes E}.$$

In order to exploit the monotonicity of L^2-norms with respect to the Kähler metrics, one uses the following trick to identify E-valued (p, q)-forms with $\tilde{E}$-valued (n, q)-forms for some holomorphic vector bundle $\tilde{E}$. We have used this trick at the bundle level, but not at the metric level.

Let $\tilde{E} := \Lambda^{p,0}T^*_X \otimes K^*_X \otimes E$ and set $\tilde{h} = g^{(p,0)} \otimes \det g \otimes h$. Then $\tilde{h}$ is a metric for $\tilde{E}$ satisfying

$$\Theta(\tilde{h})^{\sharp_{g_\varepsilon}} \geq c_\varepsilon \mathrm{Id}_{\Lambda^{n,q}_X \otimes \tilde{E}},$$

and moreover for any E-valued (p, q)-form, a.k.a. $\tilde{E}$-valued (n, q)-form, η

$$|\eta|^2_{h,g} = |\eta|^2_{\tilde{h},g}.$$

We emphasize that the metric $\tilde{h}$ involves the metric g and not the metric g_ε, and so in this sense $\tilde{h}$ is 'ε-*static*'.

Now let $\varphi \in L^2_{p,q}(g, h) = L^2_{n,q}(g, \tilde{h})$. Then by monotonicity

$$\int_X |\varphi|^2_{\tilde{h},g_\varepsilon} dV_{g_\varepsilon} \leq \int_X |\varphi|^2_{\tilde{h},g} dV_g = \int_X |\varphi|^2_{h,g} dV_g < +\infty,$$

so by Theorem 2.4.19 there exists $u_\varepsilon \in L^2_{n,q}(\tilde{h}, g_\varepsilon)$ such that

$$\bar{\partial}u_\varepsilon = \varphi \quad \text{and} \quad \int_X |u_\varepsilon|^2_{\tilde{h},g_\varepsilon} dV_{g_\varepsilon} \leq \frac{1}{c_\varepsilon}\int_X |\varphi|^2_{\tilde{h},g_\varepsilon} dV_{g_\varepsilon} \leq \frac{1}{c_\varepsilon}\int_X |\varphi|^2_{\tilde{h},g} dV_g.$$

Now, for $0 < \varepsilon < \varepsilon_o$, $g_\varepsilon \le g_{\varepsilon_o}$, and hence by monotonicity again

$$\int_X |u_\varepsilon|^2_{\tilde{h},g_{\varepsilon_o}} dV_{g_{\varepsilon_o}} \le \int_X |u_\varepsilon|^2_{\tilde{h},g_\varepsilon} dV_{g_\varepsilon} \le \frac{1}{c_\varepsilon} \int_X |\varphi|^2_{h,g} dV_g.$$

Thus since $c_\varepsilon \to c$, $\{u_\varepsilon\}$ lies in a fixed ball inside $L^2_{n,q}(\tilde{h}, g_{\varepsilon_o})$. By Alaoglu's Theorem we can choose a subsequence $u_{o,j_o} := u_{\varepsilon_{j_o}}$ that converges weakly in $L^2_{n,q}(\tilde{h}, g_{\varepsilon_o})$. Choosing $\varepsilon_1 < \varepsilon_o$, we have a weakly convergent subsequence

$$\{u_{1,j_1} \ ; \ j_1 = 1, 2, ...\} \subset \{u_{o,j_o} \ ; \ j_o = 1, 2, ...\} \cap L^2_{n,q}(\tilde{h}, g_{\varepsilon_1}).$$

Continuing inductively, we choose $\varepsilon_{k+1} < \varepsilon_k$ and a weakly convergent subsequence

$$\{u_{k+1,j} \ ; \ j = 1, 2, ...\} \subset \{u_{k,j} \ ; \ j = 1, 2, ...\} \cap L^2_{n,q}(\tilde{h}, g_{\varepsilon_{k+1}}).$$

Assuming further that $\varepsilon_k \to 0$, the sequence $\{u_j := u_{j,j}\}$ converges weakly to some

$$u \in \bigcap_{k \ge o} L^2_{n,q}(\tilde{h}, g_{\varepsilon_k}).$$

Since $|u|^2_{\tilde{h},g_{\varepsilon_k}} dV_{g_{\varepsilon_k}}$ is an increasing sequence (in k), by the Monotone Convergence Theorem

$$\int_X |u|^2_{h,g} dV_g = \int_X |u|^2_{\tilde{h},g} dV_g = \lim_{k\to\infty} \int_X |u|^2_{\tilde{h},g_{\varepsilon_k}} dV_{g_{\varepsilon_k}} \le \frac{1}{c} \int_X |\varphi|^2_{\tilde{h},g} dV_g = \frac{1}{c} \int_X |\varphi|^2_{h,g} dV_g.$$

Finally, since the convergence is weak,

$$(u, \bar{\partial}^*\psi) = \lim(u_j, \bar{\partial}^*\psi) = \lim(\varphi, \psi) = (\varphi, \psi)$$

for any compactly supported ψ. Thus $\bar{\partial} u = \varphi$, and the proof is complete. □

Weakly pseudoconvex Kähler manifolds

An important class of complete Kähler manifolds in complex analysis is the class of weakly pseudoconvex manifolds.

2.4.21 Definition. A complex manifold X is said to be weakly pseudoconvex if X has a smooth plurisubharmonic exhaustion function.

2.4.22 Theorem. *If X is a weakly pseudoconvex Hermitian (resp. Kähler) manifold then X has a complete Hermitian (resp. Kähler) metric.*

Proof. Fix a Hermitian form ω and a smooth plurisubharmonic exhaustion function ψ. Then

$$\tilde{\omega} = \omega + \sqrt{\text{-}1}\partial\bar{\partial} e^{2\psi}$$

is Hermitian, and it is Kähler if ω is Kähler. Moreover

$$\tilde{\omega} = \omega + 4e^{2\psi}\sqrt{\text{-}1}\partial\psi \wedge \bar{\partial}\psi + 2e^{2\psi}\sqrt{\text{-}1}\partial\bar{\partial}\psi \ge 4\sqrt{\text{-}1}\partial(e^\psi) \wedge \bar{\partial}(e^\psi).$$

Note that for any real-valued function ρ, $d\rho = \partial\rho + \bar{\partial}\rho = \partial\rho + \overline{\partial\rho}$, so that

$$|d\rho|^2_{\tilde{\omega}} = g^*_{\tilde{\omega}}(\partial\rho + \overline{\partial\rho}, \partial\rho + \overline{\partial\rho}) = g^*_{\tilde{\omega}}(\partial\rho, \partial\rho) + g^*_{\tilde{\omega}}(\overline{\partial\rho}, \overline{\partial\rho}) = 2g^*_{\tilde{\omega}}(\partial\rho, \partial\rho) = 2|\partial\rho|^2_{\tilde{\omega}}.$$

Thus

$$|d(e^\psi)|^2_{\tilde{\omega}} = 2|\partial(e^\psi)|^2_{\tilde{\omega}} \le \tfrac{1}{2}.$$

Since the exponential function is increasing and proper (as a function from $\mathbb{R}$ to $(0,\infty)$), e^ψ is also a smooth exhaustion function. By Proposition 2.4.15, $(X, \tilde{\omega})$ is complete. □

Thus Hörmander's Theorem holds on weakly pseudoconvex manifolds.

2.4.23 Remark. In the latter proof, if we can replace the exponential function by $\chi \circ \psi$ where $\chi : [0,\infty) \to [0,\infty)$ is any non-constant convex increasing function. (Observe that a bounded increasing convex function on $[0,\infty)$ is necessarily constant, and thus in fact $\chi \circ \psi$ is also a plurisubharmonic exhaustion.) We then compute that

$$\tilde{\omega} = \omega + \sqrt{\text{-}1}(\chi' \circ \psi)\partial\bar{\partial}\psi + \chi''(\psi)\sqrt{\text{-}1}\partial\psi \wedge \bar{\partial}\psi \ge \chi''(\psi)\sqrt{\text{-}1}\partial\psi \wedge \bar{\partial}\psi = \sqrt{\text{-}1}\partial(h \circ \psi) \wedge \bar{\partial}(h \circ \psi),$$

where h is a real-valued function satisfying $(h')^2 = \chi''$, i.e.,

$$h(x) = C + \int_o^x \sqrt{\chi''(t)}dt.$$

Note that as χ is convex, h is necessarily increasing, so in order for $h \circ \psi$ to be an exhaustion, it is necessary and sufficient that h is an exhaustion of $\mathbb{R}$, i.e., $\lim_{x\to+\infty} h(x) = +\infty$. ◇

One might wonder whether every complete Kähler manifold is weakly pseudoconvex.

2.4.24 Example. Let (X, ω) be a compact Kähler manifold and let $Y \subset X$ be a complex hypersurface. Then $Z := X - Y$ is complete Kähler. Indeed, denoting by $L_Y \to X$ the holomorphic line bundle associated to Y, let $T \in H^0(X, \mathcal{O}(L_Y))$ be a holomorphic section such that $Y = \{x \in X\ ;\ T(x) = 0\}$ and let $e^{-\lambda}$ be any smooth metric for $L_Y \to X$ such that

$$\sup_X |T|^2 e^{-\lambda} < 1.$$

The function

$$\tau := -\log|T|^2 e^{-\lambda} \in \mathscr{C}^\infty(X - Y)$$

is a negative exhaustion function of $X - Y$ that further satisfies

$$\sup_X \tau \le 0, \quad \partial\tau = \frac{\nabla^{1,0}T}{T} \quad \text{and} \quad \sqrt{\text{-}1}\partial\bar{\partial}\tau = -\sqrt{\text{-}1}\partial\bar{\partial}\lambda.$$

Note that $-\sqrt{\text{-}1}\partial\bar{\partial}\lambda$ is a smooth Hermitian $(1,1)$-form on X. Therefore for $C > 0$ sufficiently large $\sqrt{\text{-}1}\partial\bar{\partial}\tau \ge -C\omega$. Now consider the function $\rho := \phi(\tau)$, where $\phi \in \mathscr{C}^\infty(\mathbb{R})$ is a convex nondecreasing function. We compute that

$$\sqrt{\text{-}1}\partial\rho \wedge \bar{\partial}\rho = \phi'(\tau)^2\sqrt{\text{-}1}\partial\tau \wedge \bar{\partial}\tau$$

and that

$$\sqrt{\text{-}1}\partial\bar\partial\rho = \phi'(\rho)\sqrt{\text{-}1}\partial\bar\partial\tau + \phi''(\tau)\sqrt{\text{-}1}\partial\tau\wedge\bar\partial\tau.$$

Therefore the Hermitian form

$$\tilde\omega := C\omega + \sqrt{\text{-}1}\partial\bar\partial\rho = C\omega + \phi'(\tau)\sqrt{\text{-}1}\partial\bar\partial\tau + \phi''(\tau)\sqrt{\text{-}1}\partial\tau\wedge\bar\partial\tau \ge C\omega + \phi'(\tau)\sqrt{\text{-}1}\partial\bar\partial\tau$$

is a Kähler form as long as $\phi' < 1$, while

$$\tilde\omega - \sqrt{\text{-}1}\partial\rho\wedge\bar\partial\rho = C\omega + \phi'(\rho)\sqrt{\text{-}1}\partial\bar\partial\tau + (\phi''(\tau) - \phi'(t)^2)\sqrt{\text{-}1}\partial\tau\wedge\bar\partial\tau$$

shows that if $\phi' < 1$ then the Kähler form $\tilde\omega$ defines a complete metric as long as $\phi'' \ge (\phi')^2$. The function

$$\phi_o := \frac{1}{2}\left(t + \log\frac{-1}{t}\right)$$

satisfies

$$\phi_o'(t) = \frac{1}{2} - \frac{1}{2t} < \frac{1}{8} \quad \text{and} \quad \phi_o''(t) - (\phi_o'(t))^2 = \frac{1}{2t^2} + \frac{1}{2t} - \frac{1}{2} > 1$$

if $t < -\frac{4}{3}$.

On the other hand, Z need not be weakly pseudoconvex. Indeed, if X_o is any complex manifold of dimension at least 2, let X be the blowup of a point in X_o and let Y be the exceptional divisor. A plurisubharmonic function on $Z = X - Y = X_o - \{p\}$ must extend across p, and hence must be constant on X_o. Therefore such a function cannot give an exhaustion of $X - Y$. ◇

Skoda's Estimate

It is sometimes useful to replace the condition (2.16) in Hörmander's Theorem by a less uniform condition. To be more precise, let Υ be a (not necessarily positive) Hermitian $(1,1)$-form on the complete Kähler manifold X with (possibly non-complete) Kähler metric g. We say that Υ is (p,q,E)-positive if the endomorphism

$$(\Upsilon\otimes \mathrm{Id}_{\Lambda^{p,q}_X\otimes E})^{\bar\sharp_g}$$

acts positively on E-valued (p,q)-forms.

Instead of Condition (2.16), one could consider the assumption

$$\Theta(g^{(p,0)}\otimes\det g\otimes h)^{\bar\sharp_g} \ge (\Upsilon\otimes \mathrm{Id}_{\Lambda^{p,q}_X\otimes E})^{\bar\sharp_g}$$

for some (p,q,E)-positive Υ.

Such a Hermitian $(1,1)$-form Υ defines a Hermitian metric for $\Lambda^{p,q}_X\otimes E$ by the formula

$$\text{(2.17)} \qquad \langle\theta,\eta\rangle_{\Upsilon,h,g} := \left\langle(\Upsilon\otimes \mathrm{Id}_{\Lambda^{p,q}_X\otimes E})^{\bar\sharp_g}\theta,\eta\right\rangle_{g,h} = \sum_{j=1}^{q} g^{i\bar\ell}\Upsilon_{i\bar j_k}\theta^{\alpha}_{I,\bar j_1\cdots(\bar\ell)_k\cdots\bar j_q}\overline{\eta^{\beta}_{K,\bar L}}h_{\alpha\bar\beta}g^{I\bar K}g^{L\bar J},$$

where, locally, $\theta = \theta^{\alpha}_{I,\bar J}e_\alpha\otimes dz^I\wedge d\bar z^J$, $\eta = \eta^{\alpha}_{I,\bar J}e_\alpha\otimes dz^I\wedge d\bar z^J$ and $\Upsilon = \Upsilon_{i\bar j}\sqrt{\text{-}1}dz^i\wedge d\bar z^j$.

We then have the following theorem.

2.4.25 Theorem (Hörmander-Skoda-Demailly Theorem). *Let X be a complete Kähler manifold with not necessarily complete metric g and let $E \to X$ be a holomorphic vector bundle with Hermitian metric h. Assume that X admits a complete Kähler metric (which need not be g). Suppose that the curvature operator $\Theta(g^{(p)} \otimes \det g \otimes h)^{\sharp_g}$ is bounded below by a q-nonnegative Hermitian $(1,1)$-form Υ, i.e., that*

$$\Theta(g^{(p)} \otimes \det g \otimes h)^{\sharp_g} \geq (\Upsilon \otimes \mathrm{Id}_{\Lambda^{p,q}_X \otimes E})^{\bar{\sharp}_g} \geq 0 \qquad \text{on } \Lambda^{p,q}_X \otimes E.$$

Then for each Υ-nondegenerate E-valued (p,q)-form φ such that

$$\bar{\partial}\varphi = 0 \quad \text{and} \quad \int_X |\varphi|^2_{\Upsilon,h,g} dV_g < +\infty$$

there exists a E-valued $(p,q-1)$-form u such that

$$\bar{\partial}u = \varphi \quad \text{and} \int_X |u|^2_{h,g} dV_g \leq \int_X |\varphi|^2_{\Upsilon,h,g} dV_g,$$

where $|\cdot|_{\Upsilon,h,g}$ is the norm associated to the inner product (2.17).

Sketch of proof. For any smooth compactly supported E-valued (p,q)-form α we have shown that

$$\left|(\varphi,\alpha)_{g,h}\right|^2 \leq \left(\int_X |\varphi|^2_{\Upsilon,h,g} dV_g\right) \cdot \int_X \left\langle (\Upsilon \otimes \mathrm{Id}_{\Lambda^{p,q}_X \otimes E})^{\bar{\sharp}_g}\alpha, \bar{\alpha}\right\rangle_{g,h} dV_g.$$

The proof now proceeds as in that of Hörmander's Theorem. □

A final remark on regularity

An important question in applications is that of regularity of the solutions provided by Hörmander's Theorem. More precisely, if the data φ is a smooth E-valued (p,q)-form, is the form provided by Theorem 2.4.25 also smooth?

The answer to the question comes from our work on interior regularity for the Hodge Theorem. If $q = 1$ then u is a section of E, and we have

$$\Box u = \bar{\partial}^*\bar{\partial}u = \bar{\partial}^*\varphi.$$

Since the right hand side is smooth, we conclude that the section u is smooth.

If $q > 1$ the situation is a little more delicate. In fact, it is not true that if a given form u is mapped by $\bar{\partial}$ to a smooth form then u itself is smooth. The case $q = n$ gives the extreme example: $\bar{\partial}u = 0$ in the sense of distributions for absolutely any u. From this fact it follows that for every $q \geq 2$ there is a non-smooth $(p,q-1)$-form u such that $\bar{\partial}u$ is smooth.

Nevertheless, if the form φ is smooth and $\bar{\partial}$-closed, there always exists at least one smooth solution of the equation $\bar{\partial}u = \varphi$. Indeed, the Hörmander-Demailly-Skoda Theorem tells us there is some solution in $L^2_{p,q-1}(g,h)$, and provides us with an estimate. If we take the solution u_o of minimal norm, it of course satisfies the same estimate. However, since its norm is minimal, $\bar{\partial}^*u_o = 0$. (Indeed, the function

$$f(\varepsilon) := ||u_o + \varepsilon\bar{\partial}\psi||^2 = ||u_o||^2 + 2\mathrm{Re}\ \varepsilon(u_o,\bar{\partial}\psi) + \varepsilon^2||\bar{\partial}\psi||^2, \quad \varepsilon \in \mathbb{C},$$

is minimized at $\varepsilon = 0$, and thus $(u_o, \bar\partial\psi) = 0$.) Thus the minimal solution u_o satisfies

$$\Box u_o = \bar\partial^*\bar\partial u = \bar\partial^*\varphi,$$

as in the case $q = 1$. Hence u_o is smooth by the ellipticity of $\Box$.

2.5 Integrability of Involutive Structures

The principal goal of this section is to give a proof of the Newlander-Nirenberg Theorem 1.3.10 and its close cousin, Theorem 1.3.13.

Our approach to the proofs of both theorems makes crucial use of the *involutive* (as opposed to integrable) inhomogeneous Cauchy-Riemann Equations. More specifically, we shall require both L^2 estimates and regularity theorems or their solutions. We therefore begin with the study of this equation.

Since both theorems are clearly local, we work in a small neighborhood $\Omega \subset \mathbb{R}^{2n}$.

2.5.1 The Inhomogeneous Involutive Cauchy-Riemann Equations

Regularity

In fact, regularity theory for the $\bar\partial_J$-equation does not require involutivity.

Let $\omega^1, ..., \omega^n$ be a frame for $T^{*1,0}_\Omega$, and let $L_1, ..., L_n$ be the dual frame for $T^{1,0}_\Omega$. Then we can write

$$df = \sum_{j=1}^n (L_j f)\omega_j + (\bar L_j f)\bar\omega_j, \tag{2.18}$$

so that

$$\partial_J f = \sum_{j=1}^n (L_j f)\omega_j \quad \text{and} \quad \bar\partial_J f = \sum_{j=1}^n (\bar L_j f)\bar\omega_j.$$

In particular, from (2.18) we see that any first order derivative can be represented as a linear combination of the L_j and $\bar L_j$.

2.5.1 Lemma. *Let $f \in L^2(\Omega)$ have compact support in Ω and assume $\bar L_1 f, \dots, \bar L_n f \in L^2(\Omega)$. Then f is in H^1 and for any compact subset $K \subset\subset \Omega$ containing the support of f,*

$$\sum_{|\alpha|\le 1} ||D^\alpha f||_{L^2} \lesssim ||f||_{L^2} + \sum_{j=1}^n ||\bar L_j f||_{L^2}. \tag{2.19}$$

Proof. First assume $f \in \mathscr{C}^\infty_o(K)$. Denote by $\bar L_j^*$ the formal adjoint of $\bar L_j$. Then $\bar L_j^*$ is defined by

$$\int_\Omega \bar L_j^* f\bar\varphi dV = \int_\Omega f L_j \bar\varphi dV,$$

so integration by parts shows that $A := L_j + \bar{L}_j^*$ is a 0^{th} order operator. Thus

$$\begin{aligned}\int_\Omega |L_j f|^2 dV &= \int_\Omega |Af - \bar{L}_j^* f|^2 dV = \int_\Omega |Af|^2 dV + 2\mathrm{Re} \int_\Omega Af\overline{\bar{L}_j^* f} dV + \int_\Omega |\bar{L}_j^* f|^2 dV \\ &= \int_\Omega (A^*Af + 2\mathrm{Re}\ \bar{L}_j Af)\bar{f} dV + \int_\Omega (\bar{L}_j \bar{L}_j^* f)\bar{f} dV \\ &= \int_\Omega ((A^*A + 2\mathrm{Re}\ \bar{L}_j A + [\bar{L}_j, \bar{L}_j^*])f)\bar{f} dV + \int_\Omega |\bar{L}_j f|^2 dV\end{aligned}$$

Note that $(A^*A + 2\mathrm{Re}\ \bar{L}_j A + [\bar{L}_j, \bar{L}_j^*])$ is a 1^{st}-order differential operator. Thus, since any first order derivative can be represented as a linear combination of the L_j and $\bar{L}_j$, we have

$$\begin{aligned}\sum_{|\alpha|\le 1} ||D^\alpha f||_{L^2} &\lesssim \sum_{j=1}^n ||\bar{L}_j f||_{L^2}^2 + \sum_{j=1}^n ||L_j f||_{L^2}^2 \\ &\lesssim \sum_{j=1}^n ||\bar{L}_j f||_{L^2}^2 + ||f||_{L^2} \sum_{j=1}^n ||D^\alpha f||_{L^2} \\ &\lesssim \sum_{j=1}^n ||\bar{L}_j f||_{L^2}^2 + \frac{1}{\varepsilon}||f||_{L^2}^2 + \varepsilon \sum_{j=1}^n ||D^\alpha f||_{L^2}^2,\end{aligned}$$

and (2.19) follows by taking $\varepsilon > 0$ sufficiently small. If f is not smooth, we can convolve it with a mollifier to reduce the the smooth case. The proof is finished. □

As in the deduction of Theorem 2.3.12 from Gårding's Inequality 2.3.8, one proves the following result.

2.5.2 Lemma. *Let $\Omega \subset \mathbb{R}^{2n}$ be an open set and let s be some positive integer. Let $K \subset \Omega$ be a compact set and let $U \subset \Omega$ be a relatively compact open set such that $K \subset U$. If $f \in L^2(U)$ and $\bar{L}_j f \in H^s(U)$ for all $1 \le j \le n$ then*

$$\sum_{|\alpha|\le s+1} \int_K |D^\alpha f|^2 dV \lesssim \int_U |f|^2 dV + \sum_{|\alpha|\le s} \sum_{j=1}^n \int_U |D^\alpha \bar{L}_j f|^2 dV. \tag{2.20}$$

In particular, if $f \in L^2_{\ell oc}(\Omega)$ and $\bar{L}_1 f, \ldots, \bar{L}_n f \in H^s_{\ell oc}(\Omega)$ then $f \in H^{s+1}_{\ell oc}(\Omega)$.

2.5.3 Remark. By the Sobolev Embedding Theorem any f satisfying the hypotheses of Lemma 2.5.2 has continuous derivatives of order $s - n - 1$, and that

$$\sum_{|\alpha|\le 1} \sup_K |D^\alpha f|^2 dV \lesssim \int_U |f|^2 dV + \sum_{|\alpha|\le n+2} \sum_{j=1}^n \int_U |D^\alpha \bar{L}_j f|^2 dV, \tag{2.21}$$

which we deduce by taking $s = n + 2$. ◇

A priori estimates for $\bar{\partial}_J$

Unlike regularity, for the L^2 estimates of Hörmander-type one does require an involutive almost complex structure. From now on, we assume that the almost complex structure J is involutive.

Given a smooth function φ on a domain $\Omega \subset \mathbb{R}^{2n}$, we define the L^2 norm of a function u to be

$$||u||_\varphi := \left(\int_\Omega |u|^2 e^{-\varphi} dV \right)^{1/2},$$

and L^2 norm of a $(0,1)$-form $\theta = \theta_{\bar{j}} \bar{\omega}^j$ to be

$$||\theta||_\varphi := \left(\int_\Omega |\theta|^2 e^{-\varphi} dV \right)^{1/2},$$

where $|\cdot|$ in the is the Euclidean norm. We denote its associated inner product $(\theta, \theta')_\varphi$. We define $L^2(\varphi)$ to be the closure of all the smooth functions in Ω whose L^2 norm is finite. Since J is involutive, which means that

$$\bar{\partial}_J^2 = \partial_J^2 = \partial_J \bar{\partial}_J + \bar{\partial}_J \partial_J = 0,$$

all of the methods in the proof of the basic estimate go through to prove the following theorem.

2.5.4 Theorem. *Let $\Omega \subset \mathbb{R}^{2n}$ be a relatively compact open set with smooth boundary and defining function ρ, and let $\varphi \in \mathscr{C}^\infty(\overline{\Omega})$. Then for any $(0,1)$-form θ in the domains of $\bar{\partial}_J$ and $\bar{\partial}_J^*$ we have the estimate*

$$||\bar{\partial}_J^* \theta||_\Omega^2 + ||\bar{\partial}_J \theta||_Y^2 \geq \int_\Omega \theta_i \overline{\theta_j} (L_i \bar{L}_j \varphi) e^{-\varphi} dV.$$

Now consider the function $\varphi(x) = M^2 |x|^2$ for some constant $M >> 0$. Then at the origin we have

$$(L_i \bar{L}_j \varphi)(0) = M^2 \theta_i \overline{\theta_j} (L_i \bar{L}_j x_k \overline{x_k})|_{x=0} = M^2 (L_i x_k)(0) \overline{(L_j x_k)(0)} \geq \varepsilon_{ij} M^2,$$

where (ε_{ij}) is some fixed positive definite $n \times n$-matrix. Thus in a sufficiently small neighborhood of the origin, if we choose M larger than the smallest eigenvalue of (ε_{ij}) then

$$\theta_i \overline{\theta_j} (L_i \bar{L}_j \varphi) \geq M |\theta|^2.$$

Thus, perhaps again by shrinking Ω and then taking M large enough, we have the estimate

$$||\bar{\partial}_J^* \theta||_\Omega^2 + ||\bar{\partial}_J \theta||_Y^2 \geq \int_\Omega |\theta|^2 e^{-\varphi} dV.$$

for all smooth $(0,1)$-forms θ in the domain of $\bar{\partial}_J^*$. We therefore have the following Theorem.

2.5.5 THEOREM. *Let $\Omega \subset \mathbb{R}^{2n}$ be a relatively compact open set with smooth boundary and defining function ρ, and let $\varphi \in \mathscr{C}^\infty(\overline{\Omega})$. Then for any $(0,1)$-form θ in the domains of $\bar\partial_J$ and $\bar\partial_J^*$ we have the estimate*

$$||\bar\partial_J^*\theta||^2_\Omega + ||\bar\partial_J\theta||^2_Y \geq \int_\Omega |\theta|^2 e^{-\varphi} dV.$$

Consequently, one can establish Hörmander's Theorem for the operator $\bar\partial_J$. In combination with the regularity results of the previous paragraph, we have the following theorem.

2.5.6 THEOREM. *Let $\Omega \subset \mathbb{R}^{2n}$ be a sufficiently small neighborhood of the origin, and let $\varphi(x) = M|x|^2$ for some sufficiently large positive constant M. Then for any smooth $(0,1)$-form θ on Ω such that $||\theta||_\varphi < +\infty$ and $\bar\partial_J\theta = 0$ there exists a smooth function u on Ω such that*

$$\bar\partial_J u = \theta \quad \textit{and} \quad ||u||^2_\varphi \leq ||\theta||^2_\varphi.$$

The D''-equation

If one has a D''-holomorphic frame for V, i.e., a frame $\mathfrak{e}_1, ..., \mathfrak{e}_r$ such that $D''\mathfrak{e}_\nu = 0$ for every $\nu \in \{1, ..., r\}$, then the D''-equation reduces to (r copies of) the $\bar\partial_J$-equation, since in this case

$$D''(\alpha^\nu \mathfrak{e}_\nu) = (\bar\partial_J \alpha^\nu)\mathfrak{e}_\nu.$$

But if such a frame exists then Theorem 1.3.13 is already proved (see Proposition 2.5.9).

To describe the D''-equation locally we shall nevertheless fix a smooth frame $e_1, ..., e_r$. By the Liebniz rule for D''

$$D''(\alpha^i e_i) = \left(\bar\partial_J \alpha^i\right) \otimes e_i + (-1)^{\deg \alpha^i} \alpha^i \wedge D'' e_i.$$

Writing $D''e_i := A_i^j e_j$, we see that

$$D''(\alpha^i e_i) = \left(\bar\partial_J \alpha^i - (-1)^{\deg \alpha^j} A^i_j \wedge \alpha^j\right) \otimes e_i.$$

In particular, D'' differs from the (only locally defined) operator

$$\bar\partial_J(\alpha^i \otimes e_i) := \left(\bar\partial_J \alpha^i\right) \otimes e_i.$$

It is an exercise in very basic inequalities to modify what was done above for $\bar\partial_J$ so that the same results hold for D''. Leaving this exercise to the reader, we state the outcome, which is the direct analogue of Theorem 2.5.6 above.

2.5.7 THEOREM. *Let $\Omega \subset \mathbb{R}^{2n}$ be a sufficiently small neighborhood of the origin, and let $\varphi_M(x) = M|x|^2$ for some positive constant M. Define the inner products*

$$(f_1, f_2)_{\varphi_M} := \int_\Omega f_1 \cdot \bar f_2 e^{-\varphi_M} dV \quad \textit{and} \quad \left(\alpha^1, \alpha^2\right)_{\varphi_M} := \int_\Omega \delta_{i\bar j} \left\langle \alpha^i, \alpha^j \right\rangle e^{-\varphi_M} dV,$$

on $\mathbb{C}^r$-valued functions and $\mathbb{C}^r$-valued $(0,1)$-forms respectively, where $\langle\alpha,\beta\rangle$ denotes the Euclidean inner product of $\mathbb{C}$-valued 1-forms in $\mathbb{R}^{2n}$. Then there exists $M_o > 0$ such that for any $M \geq M_o$ and any smooth $\mathbb{C}^r$-valued $(0,1)$-form θ on Ω such that $||\theta||_{\varphi_M} < +\infty$ and $D''\theta = 0$ there exists a smooth $\mathbb{C}^r$-valued function u on Ω such that

$$D''u = \theta \quad \textit{and} \quad ||u||^2_{\varphi_M} \leq ||\theta||^2_{\varphi_M}.$$

2.5.2 Proof of the Newlander-Nirenberg Theorem

A 'linear' criterion for integrability

Recall that an almost complex structure J is integrable if and only if one can choose local coordinates in which J is constant (see Proposition 1.3.6). Let us state an equivalent form of this condition.

2.5.8 PROPOSITION. *Let J be an almost complex structure on a manifold X of dimension $2n$. Then J is integrable if and only if every point $p \in X$ has a neighborhood U and smooth functions $f^1, ..., f^n : U \to \mathbb{C}$ such that*

$$df^1(q) \wedge \cdots \wedge df^n \neq 0 \quad \textit{and} \quad \bar{\partial}_J f^1(q) = \cdots = \bar{\partial}_J f^n = 0 \quad \textit{at each point of } U.$$

Proof. By Proposition 1.3.6 we may assume that J is the standard almost complex structure for X. If X is a complex manifold, the needed functions are just given by the complex coordinates in any chart. In the other direction, let $F = (f^1, .., f^n)$. Then F is a local diffeomorphism, and after shrinking U we may assume F is a diffeomorphism onto its image. Finally, the complexification $dF : T_U \otimes \mathbb{C} \to T_{F(U)} \otimes \mathbb{C}$ satisfies

$$dF \circ J = \partial_J F \circ J = \sqrt{\text{-}1}\partial_J F = \sqrt{\text{-}1}dF,$$

as desired. □

We shall produce the functions $f^1, ..., f^n$ as solutions of the *linear* $\bar{\partial}_J$ equation, under the hypothesis that $\bar{\partial}_J^2 = 0$.

Conclusion of the proof

In view of Proposition 2.5.8 we must find functions $f_1, ..., f_n \in \mathrm{Ker}\bar{\partial}_J$ such that $df_1, ..., df_n$ are linearly independent in a neighborhood of the origin in $\mathbb{R}^{2n}$.

Let $g^1, ..., g^n$ be $\mathbb{C}$-valued real polynomials of degree $m+1$ on $\mathbb{R}^{2n}$ such that

$$dg^i - \omega^i \text{ vanishes to order } m \text{ at the origin}, \quad 1 \leq i \leq n.$$

Then

$$\bar{\partial}_J g^i(x) = C^i_{\bar{j}}(x)\bar{\omega}^j(x),$$

is a $\bar{\partial}_J$-closed form, where the matrix of functions $C^{i\bar{j}}(x)$ vanishes to order m at 0. We claim there for any $\varepsilon > 0$ there are functions u^i such that

$$\bar{\partial}_J u^i = \bar{\partial}_J g^i \quad \text{and} \quad |du^i(0)| < \varepsilon.$$

Indeed, by Theorem 2.5.6 there is a smooth solution u^i to the equation. Moreover, perhaps after shrinking the neighborhood Ω, (2.21) provides a solution u^i such that

$$\sup_{\Omega} |du^i| < \varepsilon$$

as soon as $m \geq 2n$. It follows that the $\mathbb{C}^r$-functions

$$f^i = g^i - u^i$$

satisfy

$$\bar{\partial}_J f^i = 0 \quad \text{and} \quad |df^i(0) - \omega^i(0)| < \varepsilon.$$

In particular, the forms $df^1, ..., df^n$ are linearly independent in a neighborhood of the origin. The Newlander-Nirenberg Theorem is thus proved. □

2.5.3 Proof of Theorem 1.3.13

Integrability for the underlying manifold: Proof of Theorem 1.3.13.i

As one might expect, Item (i) of Theorem 1.3.13 follows from the Newlander-Nirenberg Theorem. In fact, $D''D'' = 0$ implies that $\bar{\partial}_J\bar{\partial}_J = 0$. Indeed, let $p \in X$ and let $s \in \Gamma(X, \mathscr{C}^\infty(V))$ satisfy $s(p) \neq 0$. If α is a (p,q)-forms near p then

$$\begin{aligned} 0 = D''D''(\alpha \otimes s) &= D''\left(\bar{\partial}_J\alpha \otimes s + (-1)^{\deg\alpha}\alpha \wedge D''s\right) \\ &= \bar{\partial}_J\bar{\partial}_J\alpha \otimes s + (-1)^{\deg\alpha+1}\bar{\partial}_J\alpha \wedge D''s + (-1)^{\deg\alpha}\bar{\partial}_J\alpha \wedge D''s + \alpha \wedge D''D''s \\ &= \bar{\partial}_J\bar{\partial}_J\alpha \otimes s, \end{aligned}$$

and hence $\bar{\partial}_J\bar{\partial}_J\alpha = 0$ for all (p,q)-forms α.

From here on we assume that the underlying manifold X is a complex manifold, i.e., that J is the standard almost complex structure for X.

Holomorphic frame: proof of Theorem 1.3.13.ii

To complete the proof of Theorem 1.3.13 we need only find a 'holomorphic' frame in a neighborhood of every point of the base manifold X, as the next proposition shows.

2.5.9 Proposition. *Let X be a complex manifold and let $V \to X$ be smooth vector bundle with operator D'' satisfying*

$$D''\alpha \otimes s = \bar{\partial}\alpha \otimes s + (-1)^{\deg\alpha}\alpha \wedge D''s.$$

Suppose there is an open cover $\{U_j\}_{j\in J}$ for X and sections $\mathfrak{e}_{j\nu} \in \Gamma(U_j, \mathscr{C}^\infty(V))$, $1 \le \nu \le r$, such that $D''\mathfrak{e}_{j\nu} \equiv 0$ for all $1 \le \nu \le r$. Then the transition functions $g_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})$ defined by

$$(g_{ij})^\nu_\mu \mathfrak{e}_{j\nu} = \mathfrak{e}_{i\mu}$$

are holomorphic, and hence $V \to X$ is a holomorphic vector bundle.

Proof. We note simply that

$$0 = D''\mathfrak{e}_{i\mu} = D''((g_{ij})^\nu_\mu \mathfrak{e}_{j\nu}) = (\bar\partial(g_{ij})^\nu_\mu) \otimes \mathfrak{e}_{j\nu} + (g_{ij})^\nu_\mu \wedge D''\mathfrak{e}_{j\nu} = (\bar\partial(g_{ij})^\nu_\mu) \otimes \mathfrak{e}_{j\nu},$$

so that $\bar\partial(g_{ij})^\nu_\mu = 0$, as claimed. □

End of the proof of Theorem 1.3.13. Let $B \subset X$ be an open set that we identify with the unit ball in $\mathbb{C}^n$. In view of Proposition 2.5.9 we must find sections $\mathfrak{e}_1, ..., \mathfrak{e}_r \in \mathrm{Ker} D''$ that are linearly independent in a neighborhood of the origin in B.

Let $e_1, ..., e_r$ be any smooth frame for V. For each $i \in \{1, ..., r\}$ the V-valued $(0,1)$-form $\theta_i := D''e_i$ satisfies $D''\theta_i = 0$. We claim there for any $\varepsilon > 0$ there is a neighborhood $U \subset B$ of the origin and $\mathbb{C}^r$-valued functions u_i such that

$$D''u_i = \theta_i \quad \text{and} \quad \sup_U |u_i| < \varepsilon.$$

Indeed, by Theorem 2.5.7 there is a smooth solution u^i to the equation. Moreover, by (2.21)

$$\sup_U |u_i| + |du_i| < \varepsilon$$

on a suffciently small neighborhood U of the origin. It follows that the V-valued functions

$$\mathfrak{e}_i = e_i - u_i, \quad 1 \le i \le r,$$

satisfy

$$D''\mathfrak{e}^i = 0$$

and the collection $\mathfrak{e}_1, ..., \mathfrak{e}_r$ is a small perturbation of the constant functions $e_1, ..., e_r$. In particular, $\mathfrak{e}_1, ..., \mathfrak{e}_r$ is a frame. This completes the proof of Theorem 1.3.13. □

Chapter 3

Applications

3.1 Projective manifolds

Because every complex vector space (of finite dimension) admits global coordinates, it is often useful to embed a complex manifold in complex Euclidean space and exploit the ambient coordinates, or any other nice complex analytic properties of the ambient vector space.

Unfortunately, unlike the real case (proved by Whitney), not every complex manifold embeds as a submanifold of Euclidean space. Indeed, if X is a compact complex manifold of positive dimension then X cannot be embedded in a vector space, because then any linear function would restrict to the embedding as a holomorphic, hence constant, function.

The next best choice is the projectivization $\mathbb{P}(V)$ of a complex vector space V of finite dimension, since these are the simplest compact complex manifolds: they have projective coordinates, as well as a number of other very useful properties, most of which we do not discuss in these notes. However, one additional property (which is quite related to the existence of projective coordinates is important enough to discuss here.

3.1.1 Proposition. *The line bundle $\mathcal{O}(1) \to \mathbb{P}(V)$ admits a smooth Hermitian metric h of positive curvature. In particular, since $\mathcal{O}(1)$ is not trivial, the form $\sqrt{\text{-}1}\Theta(h)$ is a closed nondegenerate $(1,1)$-form that is not exact.*

Proof. After fixing a basis for V, we assume that $\mathbb{P}(V) = \mathbb{P}_n$.

The tautological line bundle $\mathcal{O}(-1) \to \mathbb{P}_n$ has a metric defined by the Euclidean metric in $\mathbb{C}^{n+1}$: if $z \in \ell$ then $z \in \mathbb{C}^{n+1}$, and square norm of z is simply its square norm as a vector in $\mathbb{C}^{n+1}$. We call this metric the *Euclidean metric* for $\mathcal{O}(-1) \to \mathbb{P}_n$.

Let us compute the curvature of the Euclidean metric. We work in the chart

$$U_o = \{[1, \zeta^1, ..., \zeta^n] \ ; \ \zeta \in \mathbb{C}^n\},$$

which we can do without loss of generality because any other chart U_j in the standard atlas for $\mathbb{P}_n$ is diffeomorphic to U_o via the diffeomorphism that swaps z^0 and z^j. The fiber over $[1, \zeta] \in U_o$ is just the set of vectors

$$\{(t, t\zeta) \in \mathbb{C} \times \mathbb{C}^n \ ; \ t \in \mathbb{C}\},$$

and the square norm is

$$|t|^2(1+|\zeta|^2) = |t|^2 e^{-\varphi_E(\zeta)}, \quad \varphi_E(\zeta) = -\log(1+|\zeta|^2).$$

The curvature of the Chern connection of the Hermitian line bundle $(\mathcal{O}(-1), e^{-\varphi_E}) \to \mathbb{P}_n$ is the $(1,1)$-form over U_o given by

$$\begin{aligned}\sqrt{\text{-}1}\partial\bar\partial\varphi_E([1,\zeta]) &= -\frac{\sqrt{\text{-}1}d\zeta\dot\wedge d\bar\zeta}{1+|\zeta|^2} + \frac{\sqrt{\text{-}1}(\bar\zeta\cdot d\zeta)\wedge(\zeta\cdot d\bar\zeta)}{(1+|\zeta|^2)^2}\\ &= -\frac{\sqrt{\text{-}1}d\zeta\dot\wedge d\bar\zeta}{(1+|\zeta|^2)^2} - \frac{|\zeta|^2\sqrt{\text{-}1}d\zeta\dot\wedge d\bar\zeta - \sqrt{\text{-}1}(\bar\zeta\cdot d\zeta)\wedge(\zeta\cdot d\bar\zeta)}{(1+|\zeta|^2)^2}.\end{aligned}$$

The first term is strictly negative, and the second term is non-positive by the Cauchy-Schwarz Inequality. Thus the dual bundle $(\mathcal{O}(1), e^{\varphi_E}) \to \mathbb{P}_n$ has positive Chern curvature. □

The dual metric e^{φ_E} is denoted $e^{-\varphi_{FS}}$, and is called the *Fubini-Study metric*. This metric assigns to a linear functional $\lambda \in \mathcal{O}(1)_{[z]}$ the squared-norm

$$|\lambda|^2 e^{-\varphi_{FS}([z])} := \frac{|\langle\lambda, z\rangle|^2}{||z||^2}.$$

The curvature

$$\sqrt{\text{-}1}\partial\bar\partial\varphi_{FS} = \frac{\sqrt{\text{-}1}d\zeta\dot\wedge d\bar\zeta}{(1+|\zeta|^2)^2} + \frac{|\zeta|^2\sqrt{\text{-}1}d\zeta\dot\wedge d\bar\zeta - \sqrt{\text{-}1}(\bar\zeta\cdot d\zeta)\wedge(\zeta\cdot d\bar\zeta)}{(1+|\zeta|^2)^2}. \tag{3.1}$$

of the Fubini-Study metric is a Kähler metric that is, unfortunately, also called the Fubini-Study metric.

3.1.2 DEFINITION. A complex manifold X is called a (complex) projective manifold if it can be embedded as a submanifold of some complex projective space $\mathbb{P}(V)$.

3.1.3 EXAMPLE. Consider the Hopf manifold $\mathrm{Hopf}_c(V)$ (c.f. Example 1.1.8). We claim that $\mathrm{Hopf}_c(V)$ is not Kähler, i.e., it does not admit a closed non-degenerate 2-form. Indeed, any form ω defines a class in $H^2_{dR}(\mathrm{Hopf}_c(V))$, but since $\mathrm{Hopf}_c(V) \cong S^1 \times S^{2n-1}$, $H^2(\mathrm{Hopf}_c(V)) = \{0\}$. Therefore $\omega = d\theta$ for some smooth 1-form θ, and we find that

$$\int_{\mathrm{Hopf}_c(V)} \omega^n = \int_{\mathrm{Hopf}_c(V)} d(\theta \wedge d(\theta^{n-1})) = 0$$

by Stokes' Theorem. But then ω^n must vanish somewhere, so ω cannot be nondegenerate.

On the other hand, $\mathbb{P}(V)$ has a closed non-degenerate 2-form, namely the curvature of $\mathcal{O}(1)$ multiplied by $\sqrt{\text{-}1}$, and hence so does any submanifold of $\mathbb{P}(V)$. ⋄

One could wonder if every Kähler submanifold is projective, but this is not the case either. We state without proof that for a generic basis $v_1, ..., v_{2n}$ of $\mathbb{R}^{2n}$, the quotient $\mathbb{C}^n/(\mathbb{Z}v_1 \oplus ... \oplus \mathbb{Z}v_{2n})$, which is a complex manifold diffeomorphic to a real torus of dimension $2n$, is Kähler but not projective.

The natural question is then to decide which compact complex manifolds are projective. The answer was provided by Kodaira, in the following result known as the Kodaira Embedding Theorem.

3.1.4 Theorem (Kodaira Embedding Theorem). *A compact complex manifold X embeds in some projective space if and only if X admits a positively curved holomorphic Hermitian line bundle.*

The principal goal of this section is to prove the Kodaira Embedding Theorem.

3.1.1 Compact complex manifolds with positive line bundles

An interpolation theorem

The most important technical tool in the basic theory of projective manifolds is the following result on finding solutions of $\bar\partial$ with prescribed vanishing at a finite number of points. This theorem will be used to establish a number of important properties of projective manifolds, including the fundamental Kodaira Embedding Theorem.

3.1.5 Theorem. *Let X be a compact complex manifold of complex dimension n and let $F \to X$ be a holomorphic line bundle with a smooth Hermitian metric $e^{-\varphi}$ whose curvature is positive. Fix a Kähler form ω on X, a holomorphic line bundle $L \to X$ with smooth Hermitian metric $e^{-\eta}$, N points $p_1, ..., p_N$, and positive integers $k_1, ..., k_N$. Then there exists an integer $m_o > 0$ with the following property. For every integer $m \ge m_o$ and every smooth $F^{\otimes m} \otimes L$-valued $(0,1)$-form α such that*

$$\bar\partial \alpha = 0, \quad \int_X |\alpha|^2_\omega e^{-(m\varphi+\eta)} dV_\omega < +\infty \quad \text{and} \quad \frac{|\alpha(z_i)|^2 e^{-(m\varphi(z_i)+\eta(z_i))}}{|z_i|^{2(n+k_i)}} \in L^1_{\ell oc}(p_i), \quad 1 \le i \le N,$$

where z_i is any local coordinate vanishing at p_i, there exists a section $u \in \Gamma(X, \mathscr{C}^\infty(F^{\otimes m} \otimes L))$ such that

$$\bar\partial u = \alpha \quad \text{and} \quad \frac{|u(z_i)|^2 e^{-(m\varphi(z_i)+\eta(z_i))}}{|z_i|^{2k_i}} \text{ vanishes at } p_i, \quad 1 \le i \le N.$$

Proof. Fix open coordinate neighborhoods $U_i = \{|z_i| < 1\}$, $1 \le i \le N$ such that

(i) the line bundles F and L are trivial on each U_i, and

(ii) for each i one has $U_i \cap \{p_1, ..., p_N\} = p_i$, and $z_i(p_i) = 0$.

Set $V_i = \{|z_i| < 1/2\}$ and fix non-negative functions $\chi_i \in \mathscr{C}^\infty_o(U_i)$ such that $\chi|_{V_i} \equiv 1$, $1 \le i \le N$. Fix an integer m and let α be an $F^{\otimes m} \otimes L$-valued $(0,1)$-form such that the assumptions of the theorem hold. For $\varepsilon \ge 0$ define the function

$$\psi_\varepsilon := \sum_{i=1}^N \chi_i \log(|z_i|^2 + \varepsilon^2)^{(n+k_i)}$$

and the metric $e^{-\varphi_{m,\varepsilon}} := e^{-(m\varphi+\eta+\psi_\varepsilon)}$ for $F^{\otimes m} \otimes L \to X$. Observe that $e^{-\varphi_{m,\varepsilon}} \nearrow e^{-\varphi_{m,o}}$ pointwise as $\varepsilon \searrow 0$.

Since ψ_ε is plurisubharmonic on $\bigcup_i V_i$ and uniformly bounded on $X - \bigcup_i V_i$, there is a constant m_o such that if $m \geq m_o$ then

$$\sqrt{-1}(\partial\bar{\partial}\varphi_{m,\varepsilon} + \mathrm{Ricci}(\omega)) \geq \omega.$$

The conditions on α in the hypotheses imply that

$$C := \int_X |\alpha|^2_\omega e^{-\varphi_{m,0}} dV_\omega < +\infty,$$

and hence

$$\int_X |\alpha|^2_\omega e^{-\varphi_{m,\varepsilon}} dV_\omega \leq C$$

for all $\varepsilon > 0$. By Hörmander's Theorem 2.4.1 there exists u_ε such that

$$\bar{\partial} u_\varepsilon = \alpha \quad \text{and} \quad \int_X |u_\varepsilon|^2 e^{-\varphi_{m,\varepsilon}} dV_\omega \leq C.$$

Since the metric $e^{-\varphi_{m,\varepsilon}}$ is a decreasing function of ε, one can use the argument in the proof of Theorem 2.4.20 to construct a measurable section u of $F^{\otimes m} \otimes L \to X$ such that $\bar{\partial} u = \alpha$ and

$$\int_X |u|^2 e^{-\varphi_{m,o}} dV_\omega < +\infty. \tag{3.2}$$

Since α is smooth, regularity of $\bar{\partial}$ for $(0,1)$-forms implies that u is also smooth. The estimate (3.2) therefore implies that u vanishes to order k_i at p_i for each i. The proof is complete. □

Theorem 3.1.5 implies a very useful interpolation theorem. To state this theorem it is convenient to introduce the notion of jets of sections. While it is possible to define k-jets in the category of $\mathscr{C}^r$ for any $k \leq r$, the holomorphic category affords us a short cut.

3.1.6 Definition (k-jets). Let $H \to X$ be a holomorphic line bundle and let $k \in \mathbb{N}$. Two germs $s, \sigma \in \mathcal{O}_{X,x}(H)$ are said to be k-equivalent if $\mathrm{Ord}_x(s - \sigma) \geq k$. An equivalence class is called a *k-jet* of sections of H at x. The set of equivalence classes is a sheaf denoted $J^k_X(H)$, and its stalks are denoted $J^k_{X,x}(H)$. We denote the projection map $\mathsf{j}^k \mathcal{O}_{X,x}(H) \to J^k_{X,x}(H)$.

If a frame ξ for H and local coordinates z near x are chosen then for any k-jet $\gamma \in J^k_{X,x}(H)$ there exists a unique polynomial $P(z) \in \mathbb{C}[z]$ of degree k such that $\gamma = \mathsf{j}^k_x(P(z)\xi)$. Thus k-jets are an invariant way of capturing the k^{th} order Taylor polynomial of a holomorphic section.

3.1.7 Theorem. *Let X be a compact complex manifold of complex dimension n and let $F \to X$ be a holomorphic line bundle with a smooth Hermitian metric $e^{-\varphi}$ whose curvature is positive. Fix a Kähler form ω on X, a holomorphic line bundle $L \to X$ with smooth Hermitian metric $e^{-\eta}$, point $x_1, ..., x_N \in X$ and integers $k_1, ..., k_N \in \mathbb{N}$. Then there is an integer m_o with the following property. For every integer $m \geq m_o$ and any finite collection of jets $\gamma_j \in J^{k_j}_{X,x_j}(F^{\otimes m} \otimes L)$, $1 \leq j \leq N$ there exists a section $s \in H^0(X, \mathcal{O}(F^{\otimes m} \otimes L))$ such that*

$$\mathsf{j}^{k_j}_x s = \gamma_j, \quad 1 \leq j \leq N. \tag{3.3}$$

Proof. Let

$$U_i := \{|z_i| < 1\}, \quad V_i := \{|z_i| < 1/2\}, \quad \chi_i \in \mathscr{C}_o^\infty(U_i), \qquad 1 \le i \le I,$$

and m_o be as in the proof of Theorem 3.1.5, applied to the set of point $x_1, ..., x_N$ and the multiplicities $k_1 + 1, ..., k_N + 1$, and fix frames ξ_i and ζ_i for $F|U_i$ and $L|_{U_i}$ respectively.

Now let $m \ge m_o$, $x_1, ..., x_N \in X$ and $\gamma_i \in J^{k_i}_{X,x_i}(F^{\otimes m} \otimes L)$, $1 \le \nu \le N$. Let P_i be the unique polynomial of degree k_i such that

$$\gamma_i = \mathrm{j}^{k_i}_{x_i}\left(P_i(z_i)\xi_i^{\otimes m} \otimes \zeta_i\right), \quad 1 \le i \le N.$$

Consider the smooth section

$$\tilde{s} := \sum_{i=1}^{N} \chi_i P_i(z_i)\xi_i^{\otimes m} \otimes \zeta_i.$$

The section $\tilde{s}$ is holomorphic on $(X - \bigcup_i U_i) \cup (\bigcup_i V_i)$, and satisfies (3.3). Therefore the $F^{\otimes m} \otimes L$-valued $(0,1)$-form $\alpha := \bar{\partial}\tilde{s}$ satisfies the hypotheses of Theorem 3.1.5, and hence there is a smooth section u that vanishes to order $k_i = \deg P_i + 1$ and satisfies $\bar{\partial}u = \alpha$. Evidently $s := \tilde{s} - u$ is the section we seek. □

Line bundles with meromorphic sections

If a holomorphic line bundle $L \to X$ has a meromorphic section s that is not identically zero then the divisor $D = \mathrm{Ord}(s)$ defines a line bundle L_D such that $s = s_D$. It follows that $L = L_D$. Therefore the question of whether every holomorphic line is of the form L_D is equivalent to the question of whether every holomorphic line bundle has a non-identically zero meromorphic section. The present paragraph gives a useful sufficient condition that guarantees a positive answer to the latter question.

3.1.8 Theorem. *Let X be a compact complex manifold. Suppose there is a holomorphic line bundle $F \to X$ with a smooth Hermitian metric $e^{-\varphi}$ whose curvature $\partial\bar{\partial}\varphi$ is strictly positive, i.e., the $(1,1)$-form $\sqrt{\text{-}1}\partial\bar{\partial}\varphi$ is a Kähler form. Then every holomorphic line bundle on X has a meromorphic section.*

Proof. Since X has a metric of strictly positive curvature, it has a closed positive $(1,1)$-form and therefore it is Kähler. We fix a Kähler form ω on X, which for instance may be taken to be $\sqrt{\text{-}1}\partial\bar{\partial}\varphi$, i.e., the Kähler form associated to the curvature form of L.

Let $L \to X$ be a holomorphic line bundle and fix $x \in X$. By Theorem 3.1.7 there exists an integer m such that the line bundles $F^{\otimes m} \otimes L \to X$ and $F^{\otimes m} \to X$ have holomorphic sections σ_1 and σ_2 with $\sigma_i(x) \neq 0$, $i = 1, 2$. Thus

$$s := \sigma_1/\sigma_2 \in \Gamma_{\mathscr{M}}(X, L) - \{0\},$$

and the proof is complete. □

Maps into projective spaces

Let X be a compact complex manifold and let $F \to X$ be a holomorphic line bundle. The vector space of global holomorphic sections $H^0(X, \mathcal{O}(F))$ consists of harmonic sections. In fact, with respect to any metric, $||\bar{\partial} s||^2 = (\Box s, s)$ and hence

$$H^0(X, \mathcal{O}(F)) = \mathscr{H}_{0,0}(X, F).$$

In particular, $H^0(X, \mathcal{O}(F))$ is a finite-dimensional vector space.

For $x \in X$, define

$$\phi_F(x) := \{s \in H^0(X, \mathcal{O}(F)) \ ; \ s(x) = 0\}.$$

If every element of $H^0(X, \mathcal{O}(F))$ vanishes at x then $\phi_F(x) = H^0(X, \mathcal{O}(F))$, and otherwise $\phi_F(x)$ is a hyperplane in $H^0(X, \mathcal{O}(F))$. Indeed, the evaluation map

$$e_x : H^0(X, \mathcal{O}(F)) \to F_x$$

has rank 0 or 1. The set of hyperplanes in $H^0(X, \mathcal{O}(F))$ is of course $\mathbb{P}(H^0(X, \mathcal{O}(F))^*)$, the projectivization of the dual space of $H^0(X, \mathcal{O}(F))$.

3.1.9 Definition. The set $\mathbb{B}\mathrm{s}(F)$ of points $x \in X$ such that $\phi_F(x) = \mathbb{P}(\Gamma_{\mathcal{O}}(X, F)^*)$ is called the (set theoretic) base locus of F. A holomorphic line bundle is said to be *free* or *base-point free* if $\mathbb{B}\mathrm{s}(F) = ‰$.

Thus we obtain a well-defined map

$$\phi_F : X - \mathbb{B}\mathrm{s}(F) \to \mathbb{P}(H^0(X, \mathcal{O}(F))^*).$$

In particular, a basepoint-free line bundle $F \to X$ results in a map of X to some projective space.

A less intrinsic way to write down the map ϕ_F is as follows. Let $s^0, ..., s^N \in H^0(X, \mathcal{O}(F))$ be a basis. We assign to $x \in X - \mathbb{B}\mathrm{s}(F)$ the point

$$[s^0(x), ..., s^N(x)] \in \mathbb{P}_N.$$

(This realization of the map $\phi_{|F|}$ shows immediately that the map is holomorphic.) To see that $[s^0, ..., s^N] : X - \mathbb{B}s(F) \to \mathbb{P}_N$ is the map $\phi_{|F|}$ expressed in projective coordinates, note that since $s^i \in H^0(X, \mathcal{O}(F)) = (H^0(X, \mathcal{O}(F))^*)^*$, the sections $s^0, ..., s^N$, when seen as a basis of vectors, serve as coordinates for $H^0(X, \mathcal{O}(F))^*$. Consider the dual basis $\{\alpha_0, ..., \alpha_k\}$ of $H^0(X, \mathcal{O}(F))^*$, i.e., $\left\langle \alpha_j, s^k \right\rangle = \delta_j^k$. A typical point $\xi \in H^0(X, \mathcal{O}(F))^*$ may then be written

$$\xi = \left\langle \xi, s^j \right\rangle \alpha_j.$$

Now, the point evaluation ξ_x at x is a linear function on $H^0(X, \mathcal{O}(F))$ with values in F_x, and thus there exist $\lambda^j(x) \in F_x$ such that

$$\xi_x = \lambda^j(x) \otimes \alpha_j.$$

But then $\lambda^j(x) = \left\langle \xi_x, s^j \right\rangle = s^j(x)$. On the other hand,

$\phi_F(x) = \{s \in H^0(X, \mathcal{O}(F))\ ;\ \xi_x s = 0\}$ corresponds to the subspace $\mathbb{C}\xi_x \in \mathbb{P}(H^0(X, \mathcal{O}(F))^*)$,

and with respect to the basis $\alpha_0, ..., \alpha_N$ of $H^0(X, \mathcal{O}(F))^*$ this subspace is

$$[s^0(x), ..., s^N(x)] \in \mathbb{P}_N.$$

More generally, instead of starting with the full vector space $H^0(X, \mathcal{O}(F))$, one can start with a subspace $V \subset H^0(X, \mathcal{O}(F))$ and define

$$\phi_{|V|}(x) := \{\sigma \in V\ ;\ \sigma(x) = 0\} \quad \text{and} \quad \mathbb{B}\mathrm{s}(V) := \{x \in X\ ;\ \sigma(x) = 0 \text{ for all } \sigma \in V\}$$

One then obtains a map

$$\phi_{|V|} : X - \mathbb{B}\mathrm{s}(V) \to \mathbb{P}(V^*).$$

In particular, a basepoint free subspace $V \subset \Gamma_{\mathcal{O}}(X, F)$ results in a holomorphic map of X to some complex projective space.

In fact, the converse is true. Indeed, suppose $\phi : X \to \mathbb{P}_N$ is a holomorphic map from a complex manifold X. Letting $F := \phi^*\mathbb{H}$ and $s^i := \phi^* z^i$, $0 \le i \le N$, we see that

$$V := \mathrm{span}_{\mathbb{C}}\{s^0, ..., s^N\} \subset \Gamma_{\mathcal{O}}(X, F) \text{ is basepoint free and } \phi = \phi_{|V|}.$$

Thus all holomorphic maps to projective space are of the form $\phi_{|V|}$ for some subspace V of holomorphic sections of a holomorphic line bundle.

3.1.10 REMARK. Note that while the sections z^i of $\mathbb{H} \to \mathbb{P}_n$ are independent, their restrictions to $\phi(X)$ need not be. Thus V is not in general isomorphic to $\mathbb{C}^{n+1}$, and may well have smaller dimension. And by the same token, there may be sections of the restriction $\mathbb{H}|_{\phi(X)} \to X$ that do not extend to holomorphic sections of $\mathbb{H} \to \mathbb{P}_n$. ◇

With our more complete understanding of projective maps, Theorem 3.1.4 is a consequence of the following more precise result.

3.1.11 THEOREM (Kodaira's Embedding Theorem Again). *Given a positive line bundle $F \to X$, there is an integer $m >> 0$ such that the map $\phi_{F^{\otimes m}}$ is an embedding.*

Properties of embeddings of compact manifolds

An embedding of a manifold M into a manifold P is by definition a proper injective immersion $F : M \to P$. If M is compact, the requirement of properness is of course vacuous.

Let $\phi_F : X \to \mathbb{P}(H^0(X, \mathcal{O}(F))^*)$ be a holomorphic map, i.e., assume that $F \to X$ is basepoint free. In order for ϕ_F to separate two distinct points $x, y \in X$, it suffices to show that there is a section $\sigma \in (H^0(X, \mathcal{O}(F))$ such that $\sigma(x) = 0 \neq \sigma(y)$. Indeed, if such a section σ exists then $\phi_F(x)$ and $\phi_F(y)$ are distinct hyperplanes $(H^0(X, \mathcal{O}(F))$, for the first contains σ and the second does not.

To prove that ϕ_F is an immersion, i.e., to show that for each $x \in X$ the linear map

$$d\phi_F(x) : T^{1,0}_{X,x} \to T^{1,0}_{\mathbb{P}((H^0(X,\mathcal{O}(F))^*),\phi_F(x)}$$

is injective, it is enough to show that for each $x \in X$ and $v \in T^{1,0}_{X,x} - \{0\}$ there exists a global section $\sigma \in (H^0(X, \mathcal{O}(F))$ such that $\sigma(x) = 0$ and $d\sigma(x)v \neq 0$. (Note that at a point where σ vanishes, $d\sigma$ is well-defined, and given by $\nabla\sigma$ for any connection ∇ for the line bundle $F \to X$.) Indeed, there exists a section $\tau \in (H^0(X, \mathcal{O}(F))$ that does not vanish at x, and if we complete the two independent sections τ and σ to a basis $\{\tau, \sigma, s^2, ..., s^N\}$ of $(H^0(X, \mathcal{O}(F))$ then in the coordinate chart

$$\mathbb{C}^N \cong_{\mathcal{O}} U_{\{\tau\neq 0\}} \subset \mathbb{P}((H^0(X, \mathcal{O}(F))^*)$$

the map ϕ_F is given by

$$G : \{y \in X \ ; \ \tau(y) \neq 0\} \ni y \mapsto \left(\tfrac{\sigma(y)}{\tau(y)}, \tfrac{s^1(y)}{\tau(y)}, ..., \tfrac{s^N(y)}{\tau(y)}\right) \in U_{\{\tau\neq 0\}},$$

and then $dG(x)v = \left(\frac{d\sigma(x)v}{\tau(x)}, *, ..., *\right) \neq 0$, i.e., $d\phi_F(x)$ is injective.

Proof of Theorem 3.1.11

We show that there is an integer $m_o > 0$ such that for all $m \geq m_o$ the map

$$\phi_{F^{\otimes m}} : X \to \mathbb{P}(H^0(X, \mathcal{O}(F^{\otimes m}))^*) \tag{3.4}$$

is an injective immersion. As just explained, we must show that for each $x \in X$ and each set of independent $\alpha^1, ..., \alpha^n \in T^{*1,0}_{X,x} \otimes F^{\otimes m}$ there are sections $\sigma_1, ..., \sigma_n \in H^0(X, \mathcal{O}(F^{\otimes m}))$, $1 \leq i \leq n$, such that

$$\sigma_i(x) = 0 \quad \text{and} \quad d\sigma_i(x) = \alpha^i, \quad 1 \leq i \leq n.$$

By Theorem 3.1.7, for each $x \in X$ there exists $m_x \in \mathbb{N}$ such that for all $m \geq m_x$ there are sections $\sigma_1, ..., \sigma_n \in H^0(X, \mathcal{O}(F^{\otimes m}))$ with these properties, i.e., such that $\phi_{F^{\otimes m}}$ is an immersion at x. Since being an immersion is an open property, there is an open cover $\{U_i\}_{i\in I}$ and positive integers $\{m_i\}_{i\in I}$ such that for each $i \in I$ the map $\phi_{F^{\otimes m}}|_{U_i}$ is an immersion for every $m \geq m_i$. Since X is compact, one can find a finite subcover $\{U_{i_1}, ..., U_{i_\nu}\}$. Taking $m_1 := \max_{1 \leq j \leq \nu} m_{i_j}$, we see that for each $m \geq m_1$ the map (3.4) is an immersion.

Next we claim that there exists m_2 such that for all $m \geq m_2$ the map (3.4) is injective. As explained earlier, it suffices to show that for each pair of distinct points $x, y \in X$ there is a section $s \in H^0(X, \mathcal{O}(F^{\otimes m}))$ such that $s(x) = 0 \neq s(y)$. We shall distinguish between pairs of points (x, y) that are 'close together' and those that are 'far apart'. To do so, consider the compact complex manifold $X \times X$, and its diagonal submanifold $\Delta_X := \{(x, x) \; ; \; x \in X\}$.

We begin by showing that if $x, y \in X$ are distinct points that are close together then for sufficiently large m (uniformly in x and y) one has $\phi_{F^{\otimes m}}(x) \neq \phi_{F^{\otimes m}}(y)$, i.e., if we define the map

$$\Phi_m : X \times X - \Delta_X \ni (x, y) \mapsto (\phi_{F^{\otimes m}}(x), \phi_{F^{\otimes m}}(y)) \in \mathbb{P}(H^0(X, \mathcal{O}(F^{\otimes m}))) \times \mathbb{P}(H^0(X, \mathcal{O}(F^{\otimes m})))$$

then $\Phi_m(y, z)$ avoids the diagonal $\Delta_{\mathbb{P}(H^0(X, \mathcal{O}(F^{\otimes m})))}$. Indeed, if $x \in X$ and $m \geq m_1$ then by the first part of the proof the map $\phi_{F^{\otimes m}}$ is an immersion near x. Hence by the Inverse Function Theorem there is a neighborhood U_x of x in X such that $\phi_{F^{\otimes m}}|_{U_x}$ is biholomorphic onto its image. In particular, $\Phi_m|_{U_x \times U_x - \Delta_X}$ avoids the diagonal $\Delta_{\mathbb{P}(H^0(X, \mathcal{O}(F^{\otimes m})))}$. Therefore, with

$$A := \bigcup_{x \in X} U_x \times U_x \subset X \times X \quad \text{and} \quad A' := A - \Delta_X \subset X \times X - \Delta_X,$$

the map $\Phi_m|_{A'}$ avoids the diagonal. Note that the open subset A' covers Δ_X, and thus the closed subset $(X \times X - \Delta_X) - A' = X \times X - A$ of $X \times X - \Delta_X$ is compact.

Next we deal with the points that are 'far apart'. By Theorem 3.1.7, for each $(x, y) \in X \times X - A$ there exists a positive integer $m_{(x,y)}$ such that if $m \geq m_{(x,y)}$ then $\phi_{F^{\otimes m}}(x) \neq \phi_{F^{\otimes m}}(y)$. Since the latter condition is open, the same constant $m_{x,y}$ will work for all $(\tilde{x}, \tilde{y})$ in an open neighborhood of (x, y) in $X \times X - \Delta_X$. Thus there is an open cover $\{U_i\}$ of $X \times X - A$ such that for each index i there is a positive integer μ_i so that if $m \geq \mu_i$ and $(x, y) \in U_i$ then $\phi_{F^{\otimes m}}(x) \neq \phi_{F^{\otimes m}}(y)$. Since $X \times X - A$ is compact, we can pass to a finite subcover $\{U_{i_1}, ..., U_{i_N}\}$. Thus, with

$$m_o := \max(m_1, \mu_{i_1}, ..., \mu_{i_N}\}$$

we find that if $m \geq m_o$ then the map $\phi_{F^{\otimes m}}$ is an embedding, as claimed. □

3.2 Stein manifolds

3.2.1 Holomorphic Convexity

3.2.1 Definition. Let Y be a complex manifold.

a. For a compact subset $K \subset Y$, the *holomorphic hull* or *$\mathcal{O}(Y)$-hull* of K is the set

$$\widehat{K} = \widehat{K}_{\mathcal{O}(Y)} := \left\{ z \in Y \; ; \; |f(z)| \leq \sup_K |f| \text{ for all } f \in \mathcal{O}(Y) \right\}.$$

b. We say Y is holomorphically convex if for every compact subset $K \subset Y$ the $\mathcal{O}(Y)$-hull $\widehat{K} \subset Y$ is also compact.

The following result, which we shall use later, is a direct analogue of Theorem **??** (and has the same proof).

3.2.2 THEOREM. *A manifold Y is holomorphically convex if and only if any sequence $\{y_j\} \subset Y$ with no accumulation point there is a subsequence $\{y_{j_k}\}$ and a function $f \in \mathcal{O}(Y)$ such that*

$$\lim_{k \to \infty} |f(y_{j_k})| = +\infty.$$

Thus a holomorphically convex manifold imposes growth conditions of sorts on its holomorphic functions. However, this growth condition does not immediately imply that there are many holomorphic functions. Indeed, one easily checks that compact complex manifolds are holomorphically convex.

Even if there are many holomorphic functions, it might happen that there are deficiencies in the algebra of global holomorphic functions. The following example is instructive.

3.2.3 EXAMPLE. Let $\Omega \subset \mathbb{C}^n$ be a domain of holomorphy containing the origin. Consider the blowup (see Example 1.1.17)

$$\tilde{\Omega}_o := \{(z, \ell) \in \Omega \times \mathbb{P}_{n-1} \ ; \ z \in \ell\}$$

of Ω at the origin, with its blowdown map $\pi : \tilde{\Omega}_o \to \Omega$ defined by $\pi(z, \ell) = z$. Note that π is proper. Indeed, the projection $\Omega \times \mathbb{P}_{n-1} \to \Omega$ is clearly proper, and the restriction of a proper map to a closed subset is proper.

Since $\pi^{-1}(0)$ is a complex submanifold biholomorphic to $\mathbb{P}_{n-1}$, every $f \in \mathcal{O}(\tilde{\Omega}_o)$ is constant on $\pi^{-1}(0)$. Thus every holomorphic function on $\tilde{\Omega}_o$ is of the form $f = \pi^* g$ for some $g \in \mathcal{O}(\Omega)$.

In particular, no $f \in \mathcal{O}(\tilde{\Omega}_o)$ can separate the points or tangents of $\pi^{-1}(0)$. On the other hand, if $K \subset \tilde{\Omega}_o$ is a compact set, then $\pi(K)$ is a compact subset of Ω. And since $\mathcal{O}(\tilde{\Omega}_o) = \pi^* \mathcal{O}(\Omega)$,

$$\widehat{K}_{\mathcal{O}(\tilde{\Omega}_o)} = \pi^{-1} \left(\widehat{\pi(K)}_{\mathcal{O}(\Omega)} \right).$$

Since π is proper, $\widehat{K}_{\mathcal{O}(\tilde{\Omega}_o)}$ is compact and therefore $\tilde{\Omega}_o$ is holomorphically convex. ◇

3.2.2 The definition of a Stein manifold

3.2.4 DEFINITION. A complex manifold Y is said to be Stein if

a. Y is holomorphically convex,

b. $\mathcal{O}(Y)$ separates points, i.e., for any distinct pair of points $x, y \in Y$ there exists $f \in \mathcal{O}(Y)$ such that $f(x) \neq f(y)$, and

c. $\mathcal{O}(Y)$ separates tangents, i.e., for any $x \in Y$ and any $\alpha \in T^{1,0}_{Y,x}$ there exists $f \in \mathcal{O}(Y)$ such that $df(x) = \alpha$.

3.2.5 Remark. Property 3 of Definition 3.2.4 is equivalent to the property that every point of Y has a coordinate chart U whose coordinates are the restrictions to U of global holomorphic functions on Y. ⋄

3.2.6 Remark. Properties 2 and 3 of Definition 3.2.4 seem rather similar in nature; if 3 holds then 2 holds at least for points x and y that are sufficiently close together, and conversely if 2 holds then at least we can find a single function with a non-zero differential. We shall see that, in fact, if Y is holomorphically convex then Property 2 implies Property 3 and vice versa. ⋄

3.2.3 A few examples

3.2.7 Example. Any holomorphically convex open subset of $\mathbb{C}^n$ is Stein. For example, $\mathbb{C}^n$ itself is Stein, and the unit ball $\mathbb{B} \subset \mathbb{C}^n$ is Stein. In fact, any convex domain $\Omega \subset \mathbb{C}^n$. ⋄

3.2.8 Proposition. *If $\iota : V \hookrightarrow Y$ is a proper holomorphic injective immersion of a complex manifold V into a Stein manifold Y then V is also a Stein manifold.*

Proof. Since the inverse image of a compact set by a proper map is by definition compact, we see that V is holomorphically convex. Next, since $\iota : V \to Y$ and $d\iota : T^{1,0}_V \to T^{1,0}_Y$ are injective maps, $\iota^* : \mathcal{O}(Y) \to \mathcal{O}(V)$ and $\iota^* : T^{*1,0}_Y \to T^{*1,0}_V$ are surjective, and thus $\iota^*\mathcal{O}(Y) \subset \mathcal{O}(V)$ separates points and tangents. The proof is complete. □

We thus find ourselves with an enormous collection of Stein manifolds.

3.2.9 Example. All closed submanifolds of $\mathbb{C}^n$ are Stein.

3.2.10 Example. Let $X \subset \mathbb{P}_N$ be a projective submanifold. Let $s \in H^0(\mathbb{P}_N, \mathcal{O}(\mathbb{H}))$ be any holomorphic section of the hyperplane line bundle $\mathbb{H} \to \mathbb{P}_N$, and let $H := \{s = 0\}$ be its zero variety. Then $Y := X - H$ is a closed submanifold of $\mathbb{P}_N - H \cong \mathbb{C}^N$. Therefore Y is a Stein manifold.

By Chow's Theorem Y is cut out by polynomials in $\mathbb{C}^N$. Such subsets of $\mathbb{C}^N$ are called *affine varieties*, or *affine manifolds* if they are also smooth. Thus a projective manifolds is a compactification of a particular kind of Stein manifold.

On the other hand, not every Stein manifold has a projective compactification. Indeed, any Stein manifold with a projective compactification must have finite topology, and there are many Stein manifolds with non-finite topology. ⋄

On the other hand, there are many important Stein manifolds that are not obviously closed complex submanifolds of some complex Euclidean space. For example, the unit ball is Stein, but it is not at all obvious that there is a closed submanifold of some complex Euclidean space that is biholomorphic to the unit ball. In fact it is not hard to show, using the maximum principle, that there is no algebraic embedding (meaning the components of the embedding are polynomials) of the ball in any complex Euclidean space. But the unit ball does embed in complex Euclidean space as a (non-affine) complex submanifold. In fact, there is a converse to Example 3.2.9:

3.2.11 Theorem. *For every Stein manifold X there exists a positive integer N and a proper holomorphic embedding $X \hookrightarrow \mathbb{C}^N$.*

Theorem 3.2.11 has some similarities with Kodaira's Embedding Theorem; one can obtain an injective holomorphic submersion $X \to \mathbb{C}^N$ for some N by using the same ideas. However, getting this map to be proper is significantly more complicated, and as a consequence we will not prove Theorem 3.2.11 in these notes.

3.2.4 The Levi Problem for Stein manifolds

The classical Levi problem is the problem of characterizing domains of holomorphy geometrically. As Stein manifolds are the manifold analogues of domains of holomorphy, the general Levi problem may be stated as follows.

3.2.12 Problem. Characterize Stein manifolds geometrically.

The goal of this paragraph is to present Grauert's solution of the Levi problem.

Strictly plurisubharmonic exhaustions on Stein manifolds

We begin with the following *'strengthening'* of Definition 2.4.21.

3.2.13 Definition. A complex manifold Y is said to be *strongly pseudoconvex* if there is a strictly plurisubharmonic exhaustion function $u \in \mathscr{C}^\infty(Y)$.

The property of being strongly pseudoconvex is a geometric property, albeit complex geometric: it says that the trivial line bundle has a positively curved metric whose weight function is moreover proper.

3.2.14 Theorem. *Let Y be a Stein manifold, K a compact subset of Y, and U a neighborhood of $\widehat{K}_{\mathcal{O}(Y)}$. Then there is a strictly plurisubharmonic exhaustion $u \in \mathscr{C}^\infty(Y)$ such that $u < 0$ on K but $u > 0$ on $Y - U$. In particular, Y is strongly pseudoconvex.*

3.2.15 Remark. In Chapter 2 we showed that every weakly pseudoconvex manifold has a complete Kähler metric (Theorem 2.4.22). Thus Theorem 3.2.14 implies that every Stein manifold is complete Kähler. ◇

Proof of Theorem 3.2.14. It suffices to assume $K = \widehat{K}_{\mathcal{O}(Y)}$. Since Y is holomorphically convex, there are compact sets $K_1 = K, K_2, ...,$ such that for all $i \geq 1$

$$K_i = (\widehat{K_i})_{\mathcal{O}(Y)}, \quad K_i \subset \text{interior}(K_{i+1}) \quad \text{and} \quad \bigcup_{i \geq 1} K_i = Y.$$

Fix open sets U_i such that $U_1 \subset U$ and $K_i \subset U_i \subset K_{i+1}$.

For each $i \geq 1$, choose functions $f_{ik} \in \mathcal{O}(Y)$, $k = 1, ..., k_i$ such that

$$\sup_{K_i} |f_{ik}| < 1 \quad \text{and} \quad \max_k |f_{ik}(z)| > 1 \text{ for all } z \in K_{i+2} - U_i.$$

By taking large powers of the f_{ik} if necessary, we can assume that

$$\sup_{K_i} \sum_{k=1}^{k_i} |f_{ik}|^2 < 2^{-i} \quad \text{and} \quad \inf_{K_{i+2}-U_i} \sum_{k=1}^{k_i} |f_{ik}|^2 > i.$$

Moreover, by the compactness of K_i and Property (3) of Stein manifolds, we can also make sure that at each point $p \in K_i$ there are n functions among the functions f_{ik}, $1 \leq k \leq k_i$, that form a local coordinate system near p.

Now set

$$u := -1 + \sum_{i=1}^{\infty} \sum_{k=1}^{k_i} |f_{ik}|^2.$$

The sum converges uniformly on each K_i, and in fact u is smooth.[1] Moreover, $u > i - 1$ on $Y - U_i$. Finally,

$$\sqrt{\text{-}1}\partial\bar{\partial}u(\xi, \xi) = \sum_{i=1}^{\infty} \sum_{k=1}^{k_i} |df_{ik}(\xi)|^2.$$

At each $x \in Y$ and each $\xi \in T_{Y,x} - \{0\}$ at least one of the summands on the right hand side is strictly positive, and therefore u is strictly plurisubharmonic. The proof is complete. □

3.2.16 Remark. We note that property (2) regarding point separation was not used in the definition of the function u above. As a consequence of Theorem 3.2.20 below, we can then deduce that on a Stein manifold Y, the point separation property of $\mathcal{O}(Y)$ is a consequence of properties (1) (holomorphic convexity) and (3) (tangent separation). ◇

Solving $\bar{\partial}$ on strongly pseudoconvex manifolds

As a warm-up to Grauert's solution of the Levi problem, we establish the following theorem.

3.2.17 Theorem. *Let X be a strongly pseudoconvex manifold of complex dimension n, let $E \to X$ be a holomorphic vector bundle and let q be a positive integer. Then for any $\bar{\partial}$-closed E-valued (p, q)-form α on X there exists a E-valued $(p, q-1)$-form u on X such that $\bar{\partial}u = \alpha$. Moreover, if α is smooth then one can choose u smooth (and if $q = 1$ all such u are smooth).*

[1] Perhaps the simplest way to see the smoothness of u is to consider the function

$$F(z, \bar{\zeta}) := -1 + \sum_{i=1}^{\infty} \sum_{k=1}^{k_i} f_{ik}(z)\overline{f_{ik}(\zeta)}$$

on the complex manifold $X \times X^\dagger$, where $X^\dagger$ is the complex manifold with the complex conjugate structure of X. For the same reasons as above, F converges locally uniformly on $X \times X^\dagger$ and is therefore holomorphic. Restricting to the real submanifold $z = \zeta$ establishes the smoothness (and even real-analyticity) of u.

Proof. By Theorem 2.4.22 X is complete Kähler. Let φ be a strictly plurisubharmonic exhaustion function on X. We take $\omega := \sqrt{\text{-}1}\partial\bar{\partial}\varphi$ as our Kähler form and denote by g the corresponding Kähler metric. If h is a smooth function of one real variable then

$$\sqrt{\text{-}1}\partial\bar{\partial}h \circ \varphi = h'(\varphi)\sqrt{\text{-}1}\partial\bar{\partial}\varphi + h''(\varphi)\sqrt{\text{-}1}\partial\varphi \wedge \bar{\partial}\varphi.$$

If we take h to be convex and sufficiently rapidly increasing then the strictly plurisubharmonic function $\psi = h \circ \varphi$ satisfies

$$\Theta(g^{(p)} \otimes \det g \otimes e^{-\psi})^{\sharp_g} \geq \omega$$

and

$$C := \int_X |\alpha|^2_\omega e^{-\psi} dV_\omega < +\infty.$$

By 2.4.20 there exists a locally integrable $(p, q-1)$-form u_o on X such that

$$\bar{\partial}u_o = \alpha \quad \text{and} \quad \int_X |u_o|^2_\omega e^{-\psi} dV_\omega < +\infty.$$

By Paragraph 2.4.2 the solution u of $\bar{\partial}u = \alpha$ with minimal norm is smooth. □

Theorem 3.2.17 yield the following outcome.

3.2.18 Corollary. *Let $E \to X$ be a holomorphic vector bundle on a Stein manifold X. Then*

$$H^{p,q}_{\bar{\partial}}(X, E) = 0 \quad \textit{for all } p \geq 0 \textit{ and } q \geq 1.$$

3.2.19 Remark. Since $\Lambda^{p,0}_X$ is a holomorphic vector bundle, the result for general p follows from the case $p = 0$. ◇

Grauert's Characterization of Stein manifolds

The following theorem of Grauert completes the solution to the Levi Problem.

3.2.20 Theorem (Grauert). *Every strongly pseudoconvex manifold is Stein.*

The proof has three steps, corresponding to the three properties in Definition 3.2.4.

Step 1: Strongly pseudoconvex manifolds are holomorphically convex

Let Y be a strongly pseudoconvex manifold and let u denote a smooth, strictly plurisubharmonic exhaustion of Y. We will prove that Y is holomorphically convex by using Theorem 3.2.2. As such, we will seek to construct a holomorphic function with prescribed values along a given sequence of points. Something similar was proved in the previous chapter, namely, Theorem 3.1.7. Of course, in the projective case we could only expect to interpolate at a finite set of points, but in the strongly pseudoconvex case the manifold carries much more positivity than can be carried by any holomorphic line bundle on a compact manifold.

Since Y has a strictly plurisubharmonic exhaustion function, it is complete Kähler. We fix a not necessarily complete Kähler form ω, and denote by dV the volume form associated to ω.

Let $\{y_j\} \subset Y$ be a closed discrete subset. We choose disjoint neighborhoods U_j in Y with biholomorphic maps $\phi_j : U_j \to \mathbb{B}$ such that $\phi_j(y_j) = 0$ and $U_i \cap \{y_j\}_{j\geq 1} = y_i$. Next choose a smooth function $\chi \in \mathscr{C}^\infty_o(\mathbb{B})$ such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in $B_{1/2}(0)$.

Observe that for all $j \geq 1$ and all $\varepsilon \in (0,1)$,

$$\text{Support}\chi(\phi_j)\log\left(|\phi_j|^2+\varepsilon^2\right) \subset U_j, \quad \chi(\phi_j)\log\left(|\phi_j|^2+\varepsilon^2\right) \leq \log 2 \text{ on } U_j$$

and

$$\left|\chi(\phi_j)\log\left(|\phi_j|^2+\varepsilon^2\right)\right| \leq \log 2 \text{ on } U_j - \phi_j^{-1}(B_{1/2}(0)). \tag{3.5}$$

Consider the $(1,1)$-form

$$\sqrt{\text{-}1}\partial\bar\partial\sum_{j=1}^{\infty} n\chi(\phi_j)\log\left(|\phi_j|^2+\varepsilon^2\right).$$

At each point of Y the sum is actually finite, and in fact consists of at most one non-zero term. Since $\log(|\cdot|^2+\varepsilon^2)$ is plurisubharmonic, it is clear that there is a smooth non-negative function $g \in \mathscr{C}^\infty(Y)$ such that

$$\sqrt{\text{-}1}\partial\bar\partial\sum_{j=1}^{\infty}\chi(\phi_j)\log\left(\frac{|\phi_j|^2+\varepsilon^2}{1+\varepsilon^2}\right)^n \geq -g\omega$$

for all $z \in Y$ and all $\varepsilon \in (0,1)$.

For an increasing convex function $h : \mathbb{R} \to (0,\infty)$ we have seen that

$$\sqrt{\text{-}1}\partial\bar\partial(h(u)) \geq h'(u)\sqrt{\text{-}1}\partial\bar\partial u.$$

By choosing h sufficiently rapidly increasing, we can guarantee that $\sqrt{\text{-}1}\partial\bar\partial h(u)$ is as positive as we like. In particular, by replacing u with $h(u)$ we can assume that

$$\sqrt{\text{-}1}\partial\bar\partial u \geq (g+1)\omega - \sqrt{\text{-}1}\text{Ricci}(\omega).$$

Thus

$$\sqrt{\text{-}1}\partial\bar\partial u + \text{Ricci}(\omega) + \sqrt{\text{-}1}\partial\bar\partial\sum_{j=1}^{\infty}\chi(\phi_j)\log\left(\frac{|\phi_j|^2+\varepsilon^2}{1+\varepsilon^2}\right)^n \geq \omega.$$

By taking h to grow even more rapidly if necessary, we can also assume that

$$\sum_{j\geq 1} j^2\int_{U_j}|\phi_j^*\bar\partial\chi|^2 e^{-u}dV < +\infty. \tag{3.6}$$

To simplify notation, let

$$\psi_\varepsilon := u + \sum_{j\geq 1}\chi(\phi_j)\log\left(\frac{|\phi_j|^2+\varepsilon^2}{1+\varepsilon^2}\right)^n.$$

Let $\tilde{f} \in \mathscr{C}^\infty(Y)$ be defined by

$$\tilde{f} := \sum_{j=1}^{\infty} j \cdot \chi(\phi_j)$$

Observe that $\tilde{f}(y_j) = j$ for all $j \geq 1$, and that $\alpha := \bar{\partial}\tilde{f} = \sum_{j=1}^{\infty} j\phi_j^* \bar{\partial}\chi$ is supported on the set

$$\bigcup_{j\geq 1} U_j - \phi_j^{-1}(B_{1/2}(0)).$$

Therefore by (3.5) and (3.6)

$$\int_Y |\alpha|^2 e^{-\psi_\varepsilon} dV \leq C_2 < +\infty,$$

where

$$C_2 = 2^n \sup_{\mathbb{B}} |\bar{\partial}\chi| \sum_{j\geq 1} j^2 \int_{U_j} e^{-u} dV.$$

In particular, C_2 is independent of ε.

By Hörmander's Theorem 2.4.20 for each $\varepsilon \in (0,1)$ there is a function $f_\varepsilon \in \mathscr{C}^\infty(Y)$ such that

$$\bar{\partial} f_\varepsilon = \alpha \quad \text{and} \quad \int_Y |f_\varepsilon|^2 e^{-\psi_\varepsilon} dV \leq C_2,$$

where C_2 is independent of ε.

Since $\log\left(\frac{|\cdot|^2+\varepsilon^2}{1+\varepsilon^2}\right)^n$ is increasing in ε, so is ψ_ε. Thus $e^{-\psi_{\varepsilon_o}} \leq e^{-\psi_\varepsilon}$ for any $\varepsilon_o > 0$ and all $\varepsilon \leq \varepsilon_o$, and therefore

$$L^2_{0,0}(e^{-\psi_\varepsilon}, \omega) \subset L^2_{0,0}(e^{-\psi_{\varepsilon_o}}, \omega) \quad \text{and} \quad \int_Y |f_\varepsilon|^2 e^{-\psi_{\varepsilon_o}} dV \leq C_2.$$

By Alaoglu's Theorem the sequence $\{f_\varepsilon\}$ is weak-* compact in each Hilbert space $L^2_{0,0}(e^{-\psi_{\varepsilon_o}}, \omega)$, and since Hilbert spaces are self-dual we can use a diagonal argument to extract a subsequence $\{f_j\} \subset \{f_\varepsilon\}$ whose limit f_o lies in each $L^2_{0,0}(e^{-\psi_\varepsilon}, \omega)$. By the Monotone Convergence Theorem $f_o \in L^2_{0,0}(e^{-\psi_o}, \omega)$, and in fact

$$\int_Y |f_o|^2 \exp\left(-\left(u + \sum_{j\geq 1} \chi(\phi_j) \log(|\phi_j|^{2n})\right)\right) dV \leq C_2 \tag{3.7}$$

(though here the precise estimate does not matter).

Now, elliptic regularity implies that f_o is smooth.[2] Since $|\phi_j|^{-2n} dV$ is not locally integrable near the origin y_j, it follows from (3.7) that $f_o(y_j) = 0$. Thus the holomorphic function $f := \tilde{f} - f_o$ satisfies

$$f(y_j) = \tilde{f}(y_j) - f_o(y_j) = j.$$

In view of Theorem 3.2.2, Y is holomorphically convex.

[2]Alternatively, $\tilde{f}$ is smooth and $f_o - \tilde{f}$ is holomorphic, so $f_o = \tilde{f} + (f_o - \tilde{f})$ is smooth.

Step 2: separating points on strongly pseudoconvex manifolds

In the previous section we showed that given a sequence $\{y_j\} \subset Y$ with no accumulation point, there is a function $f \in \mathcal{O}(Y)$ such that $f(y_j) = j$.

To separate two points x and y, we simply choose the sequence $\{y_j\} \subset Y$ such that $y_1 = x$ and $y_2 = y$. We then have a function $f \in \mathcal{O}(Y)$ such that $f(x) = 1$ and $f(y) = 2$. Thus $\mathcal{O}(Y)$ separates x and y.

Step 3: separating tangents on strongly pseudoconvex manifolds

To separate tangents is quite similar. First we note that if, in the proof of holomorphic convexity, we replace n by $n+1$ then the function f_o we obtain vanishes to second order along the points of $\{y_j\}$.

Let us, therefore, replace n by $n+1$. Now, suppose $\gamma \in T^{1,0*}_X, x$. Fix a coordinate neighborhood U of x and a biholomorphic map $\phi : U \to \mathbb{B}$ such that $\phi(x) = 0$. Choose a holomorphic function $g \in \mathcal{O}(\mathbb{B})$ such that $dg(0) = (\phi^{-1})^*\gamma$; for example, we can take $g(z) = (\phi^{-1})^*\gamma \cdot z$. Next, with χ as in the proof of holomorphic convexity above, let

$$\tilde{f} = \chi(\phi)\phi^* g.$$

Then $\tilde{f} \in \mathscr{C}^\infty_o(Y)$ is supported in U, and $d\tilde{f}(x) = \gamma$. Moreover, $\alpha := \bar{\partial}\tilde{f}$ satisfies

$$\int_Y |\alpha|^2 \exp\left(-\left(u + \chi(\phi)\log\left(\frac{|\phi|^2 + \varepsilon^2}{1+\varepsilon^2}\right)^{n+1}\right)\right) dV \le C_2 < +\infty,$$

where C_2 is independent of ε. The rest of the proof is essentially the same as the proof of holomorphic convexity, but let us give a little more of a sketch.

As in the proof of holomorphic convexity, if we replace u by $h(u)$ where h is a sufficiently rapidly increasing convex function, we have

$$\sqrt{\text{-}1}\partial\bar{\partial}\left(u + \chi(\phi)\log\left(\frac{|\phi|^2 + \varepsilon^2}{1+\varepsilon^2}\right)^{n+1}\right) + \sqrt{\text{-}1}\text{Ricci}(\omega) \ge \omega.$$

By Hörmander's Theorem and the limiting argument as above there is a function f_o such that

$$\bar{\partial} f_o = \alpha \quad \text{and} \quad \int_Y |f_o|^2 \exp\left(-\left(u + \chi(\phi)\log|\phi|^{2(n+1)}\right)\right) dV < +\infty.$$

Again $f = \tilde{f} - f_o$ is holomorphic, so $f_o \in \mathscr{C}^\infty(Y)$, and as such the integrability of f_o implies that such that

$$f_o(x) = 0 \quad \text{and} \quad df_o(x) = 0.$$

It follows in particular that

$$f := \tilde{f} - f_o \in \mathcal{O}(Y) \quad \text{and} \quad df(x) = \gamma.$$

Thus $\mathcal{O}(Y)$ separates tangents, and the proof of Grauert's Theorem 3.2.20 is complete. $\square$

Pseudoconvex domains in a Stein manifold

Let Y be a complete Kähler manifold and let Ω be an open set in Y. The domain Ω is a complex manifold in its own right, and thus it is pseudoconvex if it has a plurisubharmonic exhaustion.

3.2.21 Theorem. *If Y is a Stein manifold and $\Omega \subset\subset Y$ is pseudoconvex then Ω is a Stein manifold.*

Proof. Let $\psi \in \mathscr{C}^\infty(\Omega)$ be a plurisubharmonic exhaustion for Ω. Since Y is Stein, it has a strictly plurisubharmonic function $u \in \mathscr{C}^\infty(Y)$. But then $\psi+u$ is a strictly plurisubharmonic exhaustion for Ω. By Grauert's Theorem, Ω is Stein. □

Chapter 4

The $\mathbf{L^2}$ Extension Theorem

In this chapter we study L^2 extension of holomorphic sections from complex submanifolds.

As we know quite well by now, when working with L^2 estimates it is often convenient to work with holomorphic top forms taking values in a holomorphic line bundle, rather than just sections of a vector bundle. In that case, we have to modify what we mean by 'extension', since forms of top degree have different degrees on the submanifold and on the ambient space.

4.1 Digression: The Adjunction Formula

Let V be a smooth hypersurface in a complex manifold X. We choose an open cover $\{U_j\}$ and a family of holomorphic functions $\{f_j \in \mathcal{O}(U_j)\}$ such that $U_j \cap V = \{f_j = 0\}$ and $df_j|_{U_j \cap V}$ is never zero. The functions f_j are holomorphic because a smooth hypersurface is irreducible and the associated divisor $1 \cdot V$ has only non-negative multiplicities (all of which are 0 except the one associated to the irreducible hypersurface V, which is 1). We will abusively call V a *smooth divisor*.

The line bundle L_V given by the transition functions $g_{ij} = f_i/f_j$ therefore has a *holomorphic* section s_V defined by the local functions f_i over U_i. Differentiating the tautological identity $f_i = g_{ij} f_j$ gives

$$df_i = g_{ij} df_j + f_j dg_{ij}.$$

In particular, on V itself we have

$$df_i = g_{ij} df_j. \tag{4.1}$$

On V the differential forms df_j are not arbitrary sections of $T^*_X|_V$, but rather local sections of the so-called co-normal bundle $N^*_{X/V}$, i.e., the vector bundle whose local sections annihilate the tangent bundle of V. Since V is a hypersurface the vector bundle $N^*_{X/V} \to V$ has rank 1, i.e., it is a line bundle. Note that the dual of $N^*_{X/V}$, which is by definition the normal bundle $N_{X/V} = (T_X|_V)/T_V$, is in general not a holomorphic subbundle of $T_X|_V$ (though it can be realized as a smooth subbundle of $T_X|_V$ if some additional data is given, for example, a Hermitian metric).

The relation (4.1) thus tells us that the local sections $\{df_j\}$ piece together to form a global section ds_V of the line bundle $(L_V)|_V \otimes N^*_{X/V}$. Moreover, because V is smooth, the section ds_V is nowhere zero, and thus the line bundle $(L_V)|_V \otimes N^*_{X/V}$ is trivial. We have thus proved the following proposition.

4.1.1 PROPOSITION (Adjunction). *The line bundle $L_V \to X$ associated to a smooth divisor $V \subset X$ is an extension to X of the normal bundle $N_{X/V} \to V$.*

4.1.2 REMARK. The section ds_V can also be defined by

$$ds_V = \nabla s_V|_V,$$

where ∇ is any connection for L_V. Since any two connections differ by a global endomorphism and s_V vanishes on V, the right hand side is independent of the connection ∇. ◇

One can interpret the adjunction formula as a formula that relates the canonical bundle K_V of V to $K_X|_V$, the canonical bundle of X restricted to V. Indeed, one has the obvious formula

$$K_X|_V = K_V \otimes N^*_{X/V}, \tag{4.2}$$

which simply says that f_j locally provide coordinate functions transverse to V. Equivalently, Formula (4.2) is obtained by taking the determinant of the formula $T^{*1,0}_X|_V = T^*_V \otimes N^*_{X/V}$ (which is also obvious). Thus Proposition 4.1.1 is equivalent to the following proposition.

4.1.3 PROPOSITION (Adjunction Formula). *If $V \subset X$ is a smooth complex hypersurface then*

$$K_V = (K_X \otimes L_V)|_V.$$

4.1.4 REMARK. Given any local section f of K_V, the section

$$f \wedge ds_V$$

defines a section of $(K_X \otimes L_V)|_V$. Conversely, ds_V divides the restriction to V of any section F of $K_X \otimes L_V$.

The section f of K_V can also be computed from the section F by means of the residue theorem: F/s_V is a meromorphic top form on X, that is holomorphic except for a simple pole along V. If we choose local coordinates for X at a point of V, such that s_V is one of those coordinates and the remaining coordinates along V are z, then writing $F = \tilde{f}(z, s_V)dz^1 \wedge ... \wedge dz^{n-1} \wedge ds_V$,

$$\frac{1}{2\pi\sqrt{-1}} \int_{|s_V|=\varepsilon} \frac{F(s_V, z)}{s_V} = \tilde{f}(z, 0)dz^1 \wedge ... \wedge dz^{n-1} =: f(z).$$

Hence $F|_V = f \wedge ds_V$. ◇

4.1.5 Example. The adjunction formula is handy in an inductive proof of the very useful formula

$$K_{\mathbb{P}_n} = L_{-(n+1)H} = \mathbb{H}^{*\otimes(n+1)} \tag{4.3}$$

for the canonical bundle of projective space. Indeed, in the case $n = 1$, consider the holomorphic 1-form dz on the finite part $U_o = \{[1, z] \ ; \ z \in \mathbb{C}\}$ of $\mathbb{P}_1$. If we change charts to $U_1 = \{[w, 1] \ ; \ w \in \mathbb{C}\}$, then the coordinate change is $w = 1/z$, and we have

$$dz = -w^{-2}dw.$$

We therefore see that dz extends to a meromorphic 1-form with a double pole at $\infty = \mathbb{P}_1 - U_o$. It follows that $K_{\mathbb{P}_1}$ is the line bundle associated to the divisor -2∞, and since ∞ is just the line $w = 0$ in $\mathbb{C}^2 = \{(z, w)\}$, we see that

$$K_{\mathbb{P}_1} = L_{-2\infty} = \mathbb{H}^{*\otimes 2}.$$

Now suppose $K_{\mathbb{P}_{n-1}} = \mathbb{H}^{*\otimes n}$ has been proved. Fix a point $p \in \mathbb{P}_n$ and let H be any projective hyperplane passing through p. Since any hyperplane in H is the intersection of a hyperplane in $\mathbb{P}_n$ with $H \cong \mathbb{P}_{n-1}$, the line bundle $\mathbb{H} \to H$ used the compute $K_H \cong K_{\mathbb{P}_{n-1}}$ is the restriction to H of $\mathbb{H} \to \mathbb{P}_n$. The line bundle L_H associated to H is also $\mathbb{H}$ (see Remark **??**). Therefore by the Adjunction Formula II

$$K_{\mathbb{P}_n}|_H = (K_H \otimes L_H)|_H = \mathbb{H}^{*\otimes(n+1)}|_H,$$

and in particular, $K_{\mathbb{P}_n,p} = \mathbb{H}_p^{*\otimes(n+1)}$. Since p is arbitrary, we have (4.3) $\diamond$

The link between forms of top degree in a smooth hypersurface and those on the ambient manifold is the Adjunction Formula (see Section 4.1). Let $Z \subset X$ be a smooth hypersurface and let $T \in H^0(X, \mathcal{O}(L_Z))$ be a holomorphic section of the holomorphic line bundle associated to Z such that the divisor of T is Z, counting multiplicity. The adjunction formula tells us that $K_Z = (K_X \otimes L_Z)|_Z$. At the level of local sections, if F is a section of $K_X|_Z$ then F is divisible by dT, i.e., there exists a unique section f of K_Z such that $F|_Z = f \wedge dT$. Thus when we say that a section f on Z has an extension $\tilde{f}$ to X, we mean that

$$\tilde{f}|_Z = f \wedge dT.$$

4.2 Extension in Stein manifolds without L^2 estimates

Both for the purpose of motivation, and because we will use it later, we begin this chapter with an extension theorem that does not involve any L^2 estimates.

4.2.1 Theorem. *Let X be a Stein manifold, $Z \subset X$ a smooth hypersurface, $T \in H^0(X, \mathcal{O}(L_Z))$ a holomorphic section whose divisor is Z, and $L \to X$ a holomorphic line bundle. Then for any $f \in H^0(Z, \mathcal{O}_Z(K_Z \otimes L))$ there exists $\tilde{f} \in H^0(X, \mathcal{O}_X(K_X \otimes L_Z \otimes L))$ such that*

$$\tilde{f}|_Z = f \wedge dT.$$

We shall give two proofs of Theorem 4.2.1, which are essentially the same.

First proof. Let $\mathscr{I}_Z$ denote the coherent ideal sheaf of germs of holomorphic functions vanishing at the points of Z. Then we have the short exact sequence

$$0 \to \mathscr{I}_Z \to \mathcal{O}_X \to \mathscr{Q} \to 0.$$

The last sheaf in the sequence is the quotient $\mathscr{Q} = \mathcal{O}_X/\mathscr{I}_Z$. This sheaf is supported on Z, and its stalks at the points $z \in Z$ are $\mathcal{O}_{Z,z}$.

We can twist the sequence with the line bundle $K_X \otimes L_Z \otimes L$. Moreover, we have the adjunction isomorphism sending a germ of $K_X \otimes L_Z$ to a germ of K_Z; the map sends $F \in \mathcal{O}_{X,x}(K_X \otimes L_Z)$ to $F/dT \in \mathcal{O}_{Z,x}(K_Z)$ for any $x \in Z$. We therefore have the exact sequence of coherent sheaves

$$0 \to \mathscr{I}_Z(K_X \otimes L_Z \otimes L) \to \mathcal{O}_X(K_X \otimes L_Z \otimes L) \to \mathcal{O}_Z(K_Z \otimes L) \to 0.$$

From the induced long exact sequence in cohomology we have

$$H^0(X, \mathcal{O}_X(K_X \otimes L_Z \otimes L)) \to H^0(Z, \mathcal{O}_Z(K_Z \otimes L)) \to H^1(X, \mathscr{I}_Z(K_X \otimes L_Z \otimes L)).$$

By Cartan's Theorem B the last term is zero, and hence

$$H^0(X, K_X \otimes L_Z \otimes L) \to H^0(Z, K_Z \otimes L)$$

is surjective, as desired. □

Second proof. Fix a Kähler metric g for X, smooth metrics $e^{-\varphi_o}$ and $e^{-\lambda}$ for $L \to X$ and $L_Z \to X$, and a smooth, strictly plurisubharmonic exhaustion function u such that

$$\sqrt{\text{-}1}\partial\bar{\partial}(\varphi_o + \lambda + u) \geq \omega_g \quad \text{and} \quad \int_Z |f|^2 e^{-(\varphi_o+u)} < +\infty.$$

Such a function u can be found because X is Stein; one can take a sufficiently rapidly increasing convex function of some strictly plurisubharmonic exhaustion function on X. We shall not use the function u explicitly any more, so let us write $e^{-\varphi} := e^{-(\varphi_o+u)}$ for the metric for $L \to X$, which of course satisfies

$$\sqrt{\text{-}1}\partial\bar{\partial}(\varphi + \lambda) \geq \omega_g \quad \text{and} \quad \int_Z |f|^2 e^{-\varphi} < +\infty.$$

Consider the singular Hermitian metric for $L_Z \otimes L$ given by

$$e^{-\psi} := \frac{e^{-\varphi}}{|T|^2}.$$

Let $\{U_j\}_{j\geq 1}$ be a locally finite open cover of X by coordinate neighborhoods such that

(i) for each $j \geq 1$ the line bundles $L|_{U_j}$ and $L_Z|_{U_j}$ are trivial, and

(ii) if $U_j \cap Z \neq \emptyset$ then $U_j \cong (U_j \cap Z) \times \mathbb{D}(0, \varepsilon_j)$ is a product, and T is the holomorphic coordinate in the factor $\mathbb{D}(0, \varepsilon_j)$.

By condition (ii) we mean that on each U_j we fix trivializations ξ_j and τ_j of $L|_{U_j}$ and $L_Z|_{U_j}$, and if we write $T = T_j \tau_j$ in U_j then $|T_j(p)| < \varepsilon_j$ for each $p \in U_j$. Thus if $U_j \cap Z \neq \emptyset$ and x_j is a coordinate system for $U_j \cap Z$ then in the coordinates (x_j, T_j),

$$U_j \cong \{(x_j, T_j) \; ; \; x_j \in U_j \cap Z \text{ and } |T_j| < \varepsilon_j\}.$$

Next we define a collection of holomorphic sections $\tilde{f}_j \in H^0(U_j, \mathcal{O}_{U_j}(K_X \otimes L_Z \otimes L))$ as follows. If $U_j \cap Z = \emptyset$ then we set $\tilde{f}_j = 0$. Otherwise, we may write

$$f|_{U_j \cap Z} := f_j dx_j^1 \wedge ... \wedge dx_j^{n-1} \otimes \xi_j, \quad f_j \in \mathcal{O}(U_j \cap Z).$$

We then define $g_j \in \mathcal{O}(U_j)$ by $g_j(p) = f_j(x_j(p))$ for $p \in U_j$ and set

$$\tilde{f}_j := g_j dx_j^1 \wedge ... \wedge dx_j^{n-1} \wedge dT_j \otimes \tau_j \otimes \xi_j \in H^0(U_j, K_X \otimes L_Z \otimes L).$$

Observe that

$$\tilde{f}_j|_{Z \cap U_j} = f \wedge dT.$$

If we take ε_j sufficiently small, we also have the estimate

$$\int_{U_j} |\tilde{f}_j|^2 e^{-(\varphi+\lambda)} \leq \int_{U_j \cap Z} |f|^2 e^{-\varphi}. \tag{4.4}$$

Let us define

$$G_{ij} := \tilde{f}_i - \tilde{f}_j \in H^0(U_i \cap U_j, K_X \otimes L_Z \otimes L).$$

Then

$$G_{ij} + G_{jk} + G_{ki} = 0 \quad \text{on } U_i \cap U_j \cap U_k \quad \text{and} \quad G_{ij}|_{Z \cap U_i \cap U_j} \equiv 0.$$

We seek holomorphic sections $h_j \in H^0(U_j, K_X \otimes L_Z \otimes L)$ such that

$$G_{ij} = h_i - h_j \quad \text{and} \quad h_j|_{Z \cap U_j} \equiv 0.$$

If such functions are found then $\tilde{f}_i - h_i = \tilde{f}_j - h_j$ on $U_i \cap U_j$, and therefore

$$\tilde{f} := \tilde{f}_j - h_j \quad \text{on } U_j$$

is a globally defined holomorphic section of $K_X \otimes L_Z \otimes L \to X$ satisfying $\tilde{f}|_Z = f \wedge dT$, which is the desired conclusion. We shall now construct the sections h_j.

By using a power series argument (in the variable T_i or T_j) and estimating the Cauchy integral formula for the coefficients, we have the estimate

$$\int_{U_i \cap U_j} \frac{|G_{ij}|^2}{|T|^2} e^{-\varphi} \leq \left(2^{-j} + \int_{U_i \cap U_j \cap Z} |f|^2 e^{-\varphi}\right), \tag{4.5}$$

again provided we take $\varepsilon_j > 0$ sufficiently small. (The factor 2^{-j} is there to deal with the case in which $U_j \cap Z = \emptyset$ or $U_i \cap Z = \emptyset$ but not both.)

Next fix a partition of unity $\{\chi_j\}$ subordinate to the cover $\{U_j\}$. Define the smooth sections

$$\tilde{h}_j := \sum_k \chi_k G_{jk} \in \Gamma(U_j, \mathscr{O}_{U_j}(K_X \otimes L_Z \otimes L)).$$

Then $\tilde{h}_j$ vanish on $Z \cap U_i$. Also, by (4.4) we have

$$\int_{U_j} |\tilde{h}_j|^2 e^{-(\varphi+\lambda)} \le C_2 \left(2^{-j} + \int_{U_j \cap Z} |f|^2 e^{-\varphi} \right).$$

Now,

$$\tilde{h}_i - \tilde{h}_j = \sum_k \chi_k (G_{ik} - G_{jk}) = G_{ij} \sum_k \chi_k = G_{ij} \in H^0(U_i \cap U_j, \mathscr{O}_{U_i \cap U_j}(K_X \otimes L_Z \otimes L)),$$

and thus $\bar{\partial}\tilde{h}_i = \bar{\partial}\tilde{h}_j$ on $U_i \cap U_j$. It follows that the $L_Z \otimes L$-valued $(n,1)$-form α defined to be $\bar{\partial}\tilde{h}_i$ on U_i is well-defined, smooth and $\bar{\partial}$-closed on Ω'. Moreover, by (4.5) and the local finiteness of the cover $\{U_j\}$ we find that

$$\int_\Omega \frac{|\alpha|^2 e^{-\varphi}}{|T|^2} < +\infty.$$

(For this estimate we might have to increase the function u used to modify the metric $e^{-\varphi_o}$ at the beginning of the proof, but increasing the growth of this function only improves matters from the point of view of the Hörmander Theorem.) By Hörmander's Theorem (**??**) there exists a smooth section σ of $K_X \otimes L_Z \otimes L \to \Omega$ such that

$$\bar{\partial}\sigma = \alpha \quad \text{and} \quad \int_\Omega \frac{|\sigma|^2 e^{-\varphi}}{|T|^2} < +\infty.$$

In particular, σ vanishes on Z. Thus the sections $h_i := \tilde{h}_i - \sigma|_{U_i}$ are holomorphic, vanish on Z_j, and satisfy

$$h_i - h_j = G_{ij} \quad \text{and} \quad \int_{U_j} |h_j|^2 e^{-(\varphi+\lambda)} \le C_2 \int_{U_j \cap Z} |f|^2 e^{-\varphi}.$$

This completes the proof. □

4.3 Statement of the L^2 extension theorem

4.3.1 THEOREM. *Let X be a Stein manifold and $Z \subset X$ a smooth hypersurface. Assume there exists a holomorphic section $T \in H^0(X, \mathscr{O}_X(L_Z))$ and a Hermitian metric $e^{-\lambda}$ for L_Z such that*

$$e^{-\lambda}|_Z \not\equiv \infty \quad dT(z) \neq 0 \text{ for all } z \in Z \quad \text{and} \quad \sup_X |T|^2 e^{-\lambda} \le 1.$$

Fix a line bundle $L \to X$ and a singular Hermitian metric $e^{-\varphi}$ for L such that

$$e^{-\varphi}|_Z \not\equiv \infty, \quad \sqrt{\text{-}1}\partial\bar{\partial}\varphi \geq 0 \quad \textit{and} \quad \sqrt{\text{-}1}\partial\bar{\partial}\varphi \geq \delta\sqrt{\text{-}1}\partial\bar{\partial}\lambda$$

for some $\delta \in (0, 1]$. Then for any holomorphic section $f \in H^0(Z, \mathcal{O}_Z(K_Z \otimes L))$ such that

$$\int_Z |f|^2 e^{-\varphi} < +\infty$$

there exists a holomorphic section $F \in H^0(X, \mathcal{O}_X(K_X \otimes L_Z \otimes L))$ such that

$$F|_Z = f \wedge dT \quad \textit{and} \quad \int_X |F|^2 e^{-(\varphi+\lambda)} \leq \frac{48\pi}{\delta} \int_Z |f|^2 e^{-\varphi}.$$

Theorem 4.3.1 automatically holds for the larger class of essentially Stein manifolds, provided the hypersurface Z is compatible with the essentially Stein property. More precisely, the pair (X, Z) is said to be *essentially Stein* if there exists a hypersurface $V \subset X$ such that $X - V$ is Stein and no component of Z is contained in V.

If the hypotheses of Theorem 4.3.1 apply to X and Z then they also apply to $X' := X - V$ and $Z' := Z - V$. We then obtain, for every $f \in H^0(Z, \mathcal{O}_Z(K_{\tilde{Z}} \otimes L)) \subset H^0(Z', \mathcal{O}_{Z'}(K_{\tilde{Z}'} \otimes L))$ such that

$$\int_{Z'} |f|^2 e^{-\varphi} \leq \int_Z |f|^2 e^{-\varphi} < +\infty,$$

a holomorphic section $F \in H^0(X', K_{X'} \otimes L_{Z'} \otimes L)$ such that

$$F|_{Z'} = f \wedge dT \quad \text{and} \quad \int_{X'} |F|^2 e^{-(\varphi+\lambda)} \leq \frac{48\pi}{\delta} \int_{Z'} |f|^2 e^{-\varphi}.$$

The section F is defined on X', but we claim that it has a holomorphic extensions to X. If this is the case then one has

$$\int_X |F|^2 e^{-(\varphi+\lambda)} = \int_{X'} |F|^2 e^{-(\varphi+\lambda)}$$

because V and $V \cap Z$ are null sets in X and Z respectively.

The existence of these extensions is a consequence of the following lemma.

4.3.2 LEMMA. *Suppose Y is a complex manifold with a holomorphic line bundle $L \to Y$ having a singular Hermitian metric $e^{-\psi}$ whose curvature current is bounded below by a smooth Hermitian $(1,1)$-form, that $W \subset Y$ is a (possibly singular) subvariety such that $e^{-\psi}|_W \not\equiv \infty$, and that $g \in H^0(Y - W, \mathcal{O}_{Y-W}(K_Y \otimes L))$ is a holomorphic section such that*

$$\int_{Y-W} |g|^2 e^{-\psi} < +\infty.$$

Then there exists a unique holomorphic section $\tilde{g} \in H^0(Y, \mathcal{O}_Y(K_Y \otimes L))$ such that

$$\tilde{g}|_{Y-W} = g.$$

Proof. By the Identity Principle, the extension problem is local.

If W is singular then the singular locus $W_{\rm sing}$ of W is a complex subvariety of complex codimension at least 2. If we have proved that the section g can be extended holomorphically across the smooth locus $W_{\rm reg}$ of W, then g can also be extended holomorphically across $W_{\rm sing}$ by Hartogs' Theorem. Thus it suffices to prove that g has a holomorphic extension across $W_{\rm reg}$. Moreover, by slicing with an appropriately generic hyperplane we may assume that W is a hypersurface.

Let $p \in W_{\rm reg}$ and fix a relatively compact (in Y) coordinate neighborhood U of p in Y with coordinates $z = (z^1, ..., z^{n-1}, z^n)$ the following properties:

(i) $L|_U$ is trivial (with frame ξ, say),

(ii) $U \cap W_{\rm sing} = \emptyset$ and $z^i|_{U\cap W}$, $1 \le i \le n-1$ are local coordinates on $W_{\rm reg}$, and

(iii) $z(x) = 0$ and $W \cap U = \{z^n = 0\}$.

Our hypothesis on $e^{-\psi}$ implies that $-\log|\xi|^2 e^{-\psi}$ is quasi-plurisubharmonic, hence locally bounded above. In $U - W$ we may write

$$g = h(z) dz^1 \wedge ... \wedge dz^n \otimes \xi$$

where $h \in \mathcal{O}(U - W)$. Let $B = \{y \in U \ ; \ |z^i(y)| < 1 \text{ for } 1 \le i \le n\}$ Note that

$$\int_{B-W} |h(z)|^2 dV(z) \le \sup_B (|\xi|^2 e^{-\psi})^{-1} \int_{B-W} |g|^2 e^{-\psi} < +\infty.$$

Writing $w = (z^1, ..., z^{n-1})$, we have the Laurent expansion

$$h(z) = \sum_{j\in\mathbb{Z}} A_j(w)(z^n)^j,$$

where for each $j \in \mathbb{Z}$ the function A_j is holomorphic on the unit polydisk $\mathbb{D}^{n-1}$. Then we have

$$\int_{B-W} |h(z)|^2 dV(z) = \lim_{\varepsilon\to 0} \int_{\mathbb{D}^{n-1}} \int_{\varepsilon \le |z^n| < 1} \sum_{j,k\in\mathbb{Z}} A_k(w)\overline{A_k(w)}(z^n)^j (\bar{z}^n)^k dV(w) dA(z^n).$$

Now,

$$\begin{aligned}
&\int_{\mathbb{D}^{n-1}} \int_{\varepsilon \le |z^n| < 1} \sum_{j,k\in\mathbb{Z}} A_j(w)\overline{A_k(w)}(z^n)^j (\bar{z}^n)^k dV(w) dA(z^n) \\
&= \sum_{j,k\in\mathbb{Z}} \left(\int_{\mathbb{D}^{n-1}} A_j(w)\overline{A_k(w)} dV(w) \right) \int_{\varepsilon \le |z^n| < 1} (z^n)^j (\bar{z}^n)^k dA(z^n) \\
&= \left(\log \tfrac{1}{\varepsilon}\right) \int_{\mathbb{D}^{n-1}} |A_{-1}(w)|^2 dV(w) + \sum_{j\in\mathbb{Z}-\{-1\}} \frac{\pi(1-\varepsilon^{2j+2})}{2j+2} \int_{\mathbb{D}^{n-1}} \left|A_j(w)\right|^2 dV(w).
\end{aligned}$$

The latter is bounded as $\varepsilon \to 0$ if and only if $A_j = 0$ for all $j \le -1$. Thus

$$h(z) = \sum_{j\in\mathbb{N}} A_j(w)(z^n)^j,$$

which shows that h has a holomorphic extension to B, as needed. □

4.4 Some Corollaries of the Extension Theorem

4.4.1 Extension from a hyperplane to a domain in $\mathbb{C}^n$

The first version of the L^2 extension theorem was proved by Ohsawa and Takegoshi. They proved a result that looks rather different than our Theorem 4.3.1, but both theorems imply the following corollary, which was also stated in the article [**?**] of Ohsawa and Takegoshi.

4.4.1 Theorem (Ohsawa-Takegoshi). *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, let $\varphi \in \mathrm{PSH}(\Omega)$ and let H be an affine hyperplane in $\mathbb{C}^n$. Assume that $\varphi|_{H\cap\Omega} \not\equiv -\infty$. Then there is a constant C depending only on the diameter of Ω such that for any $f \in \mathcal{O}(H\cap\Omega)$ satisfying*

$$\int_{H\cap\Omega} |f|^2 e^{-\varphi} dA < +\infty$$

there exists $F \in \mathcal{O}(\Omega)$ such that

$$F|_{H\cap\Omega} = f \quad \textit{and} \quad \int_\Omega |F|^2 e^{-\varphi} dV \le C \int_{H\cap\Omega} |f|^2 e^{-\varphi} dA.$$

Proof. Choose coordinates in $\mathbb{C}^n$ such that $H = \{z^n = 0\}$ and let

$$D_H := \sup_\Omega |z^n|.$$

Then $D_H < +\infty$ because Ω is bounded. (Note that the finiteness of D_H does not require Ω to be bounded, but only that Ω be contained between two real hyperplanes $\mathfrak{H}_1$ and $\mathfrak{H}_2$ such that $\mathfrak{H}_1 \cap \sqrt{\text{-}1}\mathfrak{H}_1$ and $\mathfrak{H}_2 \cap \sqrt{\text{-}1}\mathfrak{H}_2$ are parallel to H. We call D_H the H-diameter of Ω.) The line bundle associated to H is of course trivial, and its canonical holomorphic section is the coordinate function $T = z^n$. We equip this line bundle with the metric (i.e., positive function)

$$e^{-\lambda} = \frac{1}{D_H^2}.$$

Thus $\sup_\Omega |T|^2 e^{-\lambda} = 1$. Moreover, $\partial\bar\partial\lambda = 0$, and since φ is plurisubharmonic,

$$\sqrt{\text{-}1}\partial\bar\partial\varphi \ge 0 \quad \text{and} \sqrt{\text{-}1}\partial\bar\partial\varphi \ge \sqrt{\text{-}1}\partial\bar\partial\lambda.$$

Thus Theorem 4.3.1 applies, with $\delta = 1$.

To f we assign the holomorphic $(n-1)$-form $\alpha = f dz^1 \wedge \ldots \wedge dz^{n-1}$, so that

$$\int_{\Omega\cap H} |\alpha|^2 e^{-\varphi} = 2^{1-n} \int_{H\cap\Omega} |f|^2 e^{-\varphi} dA < +\infty.$$

By Theorem 4.3.1 there is a holomorphic n-form $\beta = F dz^1 \wedge \ldots \wedge dz^n$ such that

$$\beta|_H = \alpha \wedge dz^n, \quad \text{i.e.,} \quad F|_{H\cap\Omega} = f$$

and

$$\frac{2^n}{D_H^2} \int_\Omega |F|^2 e^{-\varphi} dV = \int_\Omega |\beta|^2 e^{-(\varphi+\lambda)} \le 2^n \cdot 16\pi \int_{H\cap\Omega} |f|^2 e^{-\varphi} dA.$$

Thus $C = 16\pi D_H^2$, and the proof is complete. □

4.4.2 Remark. In recent years there has been interest in finding the optimal constant, i.e., the smallest constant C for which Theorem 4.4.1 holds independent of the choice of plurisubharmonic weight function φ. The optimal constant was found by Z. Błocki, who showed that one can take $C = \pi D_H^2$. ◇

A very useful corollary of Theorem 4.4.1 is obtained by applying the theorem inductively.

4.4.3 Corollary. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex domain and let $\varphi \in \mathrm{PSH}(\Omega)$ such that $\varphi(p) \neq -\infty$. Then there is a constant C depending only on the diameter of Ω such that for any $a \in \mathbb{C}$ there exists $f \in \mathcal{O}(\Omega)$ satisfying*

$$f(p) = a \quad \textit{and} \quad \int_\Omega |f|^2 e^{-\varphi} dV \leq C|a|^2 e^{-\varphi(p)}.$$

4.4.2 An extension theorem in $\mathbb{C}^n$

Another special case of Theorem 4.3.1 is an extension result that appears in the theory of interpolation. We begin with a smooth hypersurface $Z \subset \mathbb{C}^n$, and a smooth Hermitian metric $e^{-\varphi}$ on $\mathbb{C}^n$ for the trivial line bundle. Denote the Euclidean metric in $\mathbb{C}^n$ by $\omega := \sqrt{\text{-}1}\partial\bar\partial|z|^2$. Let $T \in \mathcal{O}(\mathbb{C}^n)$ be a holomorphic function such that

$$Z = \{T = 0\} \quad \text{and} \quad dT(x) \neq 0 \quad \text{for all } x \in Z.$$

We define the metric $e^{-\lambda_r^T}$ for the trivial bundle using T itself:

$$\lambda_r^T(x) := \fint_{B(x,r)} \log |T(\zeta)|^2 \omega^n(\zeta).$$

(The notation is $\fint_X f d\mu := \frac{1}{\mu(X)} \int f d\mu$.) We have the following result.

4.4.4 Theorem. *Assume there exists $\delta > 0$ such that*

$$\sqrt{\text{-}1}\partial\bar\partial\varphi \geq (1+\delta)\sqrt{\text{-}1}\partial\bar\partial\lambda_r^T. \tag{4.6}$$

Then for every holomorphic function $f \in \mathcal{O}(Z)$ such that

$$\int_Z \frac{|f|^2 e^{-\varphi}\omega^{n-1}}{|dT|^2 e^{-\lambda_r^T}} < +\infty$$

there exists $F \in \mathcal{O}(\mathbb{C}^n)$ such that

$$F|_Z = f \quad \textit{and} \quad \int_{\mathbb{C}^n} |F|^2 e^{-\varphi}\frac{\omega^n}{n!} \leq 48\pi \int_Z \frac{|f|^2 e^{-\varphi}}{|dT|^2 e^{-\lambda_r^T}} \frac{\omega^{n-1}}{(n-1)!}. \tag{4.7}$$

Proof. The line bundle L_Z associated to Z is trivial, since it is a line bundle on $\mathbb{C}^n$. Of course, it has the non-trivial (meaning, non-flat) metric $e^{-\lambda_r^T}$. The sub-mean value property for the plurisubharmonic function $\log|T|$ means that

$$|T|^2 e^{-\lambda_r^T} \leq 1.$$

Moreover, since

$$\left(\frac{\sqrt{\text{-}1}}{2\pi}\partial\bar{\partial}\lambda_r^T\right)(x) = \left(\frac{1}{2\pi}\fint_{B(x,r)} \frac{\partial^2}{\partial\zeta^i\partial\bar{\zeta}^j}\log|T(\zeta)|^2\omega(\zeta)\right)\sqrt{\text{-}1}dx^i \wedge d\bar{x}^j,$$

λ_r^T is plurisubharmonic, and thus (4.6) implies the curvature conditions required in Theorem 4.3.1.

Let f be the function to be extended. Consider the holomorphic n-form $g := f dz^1 \wedge ... \wedge dz^n$ defined along the points of Z. (Thus g is a section of $K_{\mathbb{C}^n}$ along the points of Z. But it is not a holomorphic n-form on Z because Z has dimension $n-1$, so $(n,0)$-form on Z is identically zero.) Since the 1-form dT is nowhere-zero on Z, there exists a holomorphic form h of top degree on Z, i.e., a holomorphic section of $K_Z \to Z$, such that

$$g = h \wedge dT \quad \text{along } Z.$$

Suppose, for a moment, that

$$\int_Z |h|^2 e^{-\varphi} < +\infty.$$

Then by Theorem 4.3.1, applied to the metric $\psi = \varphi - \lambda_r^T$, there exists a holomorphic n-form H on $\mathbb{C}^n$ such that

$$H|_Z = h \wedge dT \quad \text{and} \quad \int_{\mathbb{C}^n} |H|^2 e^{-\varphi} \le 48\pi \int_Z |h|^2 e^{-\varphi + \lambda_r^T}.$$

Now, we can write $H = F dz^1 \wedge ... \wedge dz^n$, and since $h \wedge dT = g = f dz^1 \wedge ... \wedge dz^n$, we see that

$$F|_Z = f \quad \text{and} \quad \int_{\mathbb{C}^n} |H|^2 e^{-\varphi} = \int_{\mathbb{C}^n} |F|^2 e^{-\varphi}\frac{\omega^n}{n!}$$

Thus our proof will be complete if we show that

$$\iota_Z^*|h|^2 := \iota_Z^*\left(\sqrt{\text{-}1}^{n^2} h \wedge \bar{h}\right) = \frac{|f|^2}{|dT|^2}\frac{\iota_Z^*\omega^{n-1}}{(n-1)!},$$

where $\iota_Z : Z \hookrightarrow \mathbb{C}^n$ is the inclusion. To establish this identity, we can work at a single point $x \in Z$. After a translation and a unitary change of coordinates (both of which leave ω invariant) we may assume $x = o$, that $z^1, ..., z^{n-1}$ are coordinates on Z near o, and that

$$z^n - \frac{T}{|dT(o)|}$$

vanishes to second order at o. Therefore, at the origin,

$$g(o) = \frac{f(o)}{|dT(o)|}(dz^1 \wedge ... \wedge dz^{n-1} \wedge dT)(o).$$

Writing $h = h_o dz^1 \wedge ... \wedge dz^{n-1}$ near the origin, we see that

$$h_o(o) = f(o)/|dT(0)|.$$

Since

$$(\iota_Z^* h \wedge \overline{h})(o) = |h_o(o)|^2 \frac{\iota_Z^*\omega^{n-1}(o)}{(n-1)!},$$

our claim follows, and the proof is complete. □

4.5 Proof of the L^2 extension theorem

At various stages of our proof we will need to work on relatively compact subdomains of X. Since we have assumed that X is Stein, there is an exhaustion of X be relatively compact pseudoconvex domains. We fix such a domain $\Omega \subset\subset X$ in much of the proof. In the end we will take a limit, in an appropriate sense, as $\Omega \nearrow X$.

We also assume that the metrics $e^{-\varphi}$ and $e^{-\lambda}$ are smooth. Indeed, one can reduce to the case of smooth metrics by the usual weak limit arguments and the fact that on Stein manifolds one can approximate such metrics locally uniformly by a monotonic sequence of smooth metrics.

4.5.1 Twisted Basic Estimate

In the present paragraph we shall be dealing with two smooth metrics $e^{-\varphi}$ and $e^{-\psi}$. In order to keep things maximally clear, we shall denote the (either formal or Hilbert space) adjoints of $\bar\partial$ with respect to these metrics as $\bar\partial^*_\varphi$ and $\bar\partial^*_\psi$ respectively.

4.5.1 LEMMA. *Let Ω be a pseudoconvex domain in a Stein manifold X of complex dimension n, and let $H \to X$ be a holomorphic line bundle with smooth Hermitian metric $e^{-\psi}$. Fix a Kähler metric g on X. Let τ and A be positive functions on Ω, with $\tau \in \mathscr{C}^2(\Omega)$. Then for any H-valued $(n,1)$-form $u \in$ Domain$(\bar\partial) \cap$ Domain$(\bar\partial^*_\psi)$ on Ω, the following inequality holds.*

$$\begin{aligned}(4.8)\qquad &\int_\Omega (\tau + A)\left|\bar\partial^*_\psi u\right|^2 e^{-\psi} + \int_\Omega \tau \left|\bar\partial u\right|^2_g e^{-\psi} \\ &\ge \int_\Omega \left(\tau\partial\bar\partial\psi - \partial\bar\partial\tau - \frac{1}{A}\partial\tau\wedge\bar\partial\tau\right)_{i\bar j} g^{i\bar k} u_{\bar k}\overline{g^{j\bar\ell}u_{\bar\ell}} e^{-\psi}.\end{aligned}$$

Proof. Let $e^{-\kappa}$ be a smooth Hermitian metric for $H \to X$. In the present setting we have the following special case of the basic estimate:

$$(4.9)\qquad \int_\Omega \left|\bar\partial^*_\kappa u\right|^2 e^{-\kappa} + \int_\Omega \left|\bar\partial u\right|^2_g e^{-\kappa} \ge \int_\Omega (\partial\bar\partial\kappa)_{i\bar j} g^{i\bar k} u_{\bar k}\overline{g^{j\bar\ell}u_{\bar\ell}} e^{-\kappa}$$

for any H-valued $(n,1)$-form $u \in$ Domain$(\bar\partial^*_\kappa) \cap$ Domain$(\bar\partial)$. Note that the last term on the right hand side of (4.9) is non-negative.

Next we pass the the twisted estimates. The idea is to consider a twist of the original metric for H. That is to say, we consider a metric $e^{-\psi}$ for H. For any such metric, there exists a positive function τ such that

$$e^{-\kappa} = \tau e^{-\psi}.$$

One calculates that

$$\partial\bar\partial\kappa = \partial\bar\partial\psi - \tau^{-1}\partial\bar\partial\tau + \tau^{-2}\partial\tau\wedge\bar\partial\tau.$$

Also, using the formula

$$\bar\partial^*_\kappa u = -g^{i\bar j} e^{\kappa}\frac{\partial}{\partial z^i}(e^{-\kappa}u_{\bar j})$$

for the formal adjoint, where locally $u = \sum u_{\bar{j}} d\bar{z}^j$ with $u_{\bar{j}}$ canonical sections, we have the formula

$$\bar{\partial}^*_\kappa u = -\tau^{-1} g^{i\bar{j}} \frac{\partial \tau}{\partial z^i} u_{\bar{j}} + \bar{\partial}^*_\psi u.$$

Substitution of these identities into the basic estimate (4.9) then gives

$$\begin{aligned} &\int_\Omega (\tau + A) \left|\bar{\partial}^*_\psi u\right|^2 e^{-\psi} + \int_\Omega \tau \left|\bar{\partial} u\right|^2_g e^{-\psi} \\ &\geq -\int_\Omega (\partial\bar{\partial}\tau)_{i\bar{j}} g^{i\bar{k}} u_{\bar{k}} \overline{g^{j\bar{\ell}} u_{\bar{\ell}}} e^{-\psi} + \int_\Omega (\tau \partial\bar{\partial}\psi)_{i\bar{j}} g^{i\bar{k}} u_{\bar{k}} \overline{g^{j\bar{\ell}} u_{\bar{\ell}}} e^{-\psi} \\ &\quad + \int_\Omega A \left|\bar{\partial}^*_\psi u\right|^2 e^{-\psi} + 2\mathrm{Re} \int_\Omega g^{i\bar{j}} \frac{\partial \tau}{\partial z^i} u_{\bar{j}} \overline{\bar{\partial}^*_\psi u} e^{-\psi}. \end{aligned} \tag{4.10}$$

By the Cauchy-Schwarz Inequality,

$$2\mathrm{Re} \int_\Omega g^{i\bar{j}} \frac{\partial \tau}{\partial z^i} u_{\bar{j}} \overline{\bar{\partial}^*_\psi u} e^{-\psi} \geq -\int_\Omega A \left|\bar{\partial}^*_\psi u\right|^2 e^{-\psi} - \int_\Omega \frac{(\partial\tau \wedge \bar{\partial}\tau)_{i\bar{j}}}{A} g^{i\bar{k}} u_{\bar{k}} \overline{g^{j\bar{\ell}} u_{\bar{\ell}}} e^{-\psi}.$$

Applying this estimate to the last term on the third line of (4.10) completes the proof. □

4.5.2 Choices for τ, A and ψ, and an a priori estimate

In Lemma 4.5.1 we let $H := L_Z \otimes L$. Note that

$$\partial\bar{\partial} \log |T|^2 e^{-\lambda} = \partial\bar{\partial} \log |T|^2 - \partial\bar{\partial}\lambda$$

in the sense of currents. The current $\partial\bar{\partial}\log|T|^2$ is supported on Z because T is zero-free on $X - Z$. (In fact, this current is a multiple of the current of integration over Z, a statement that we will prove later, but that we do not currently require.) We define the quasi-plurisubharmonic function

$$v = \log |T|^2 e^{-\lambda}.$$

Fix a constant $\gamma > 1$. (Eventually we will let $\gamma \to 1$.) Define the function

$$a := \gamma - \delta \log(e^v + \varepsilon^2). \tag{4.11}$$

Note that $a > 1$ if $\varepsilon > 0$ is sufficiently small. We calculate that

$$\begin{aligned} -\partial\bar{\partial} a = \delta\partial\bar{\partial}\log(e^v + \varepsilon^2) &= \partial\left(\frac{\delta e^v \bar{\partial} v}{e^v + \varepsilon^2}\right) \\ &= \frac{\delta e^v}{(e^v + \varepsilon^2)} \partial\bar{\partial} v + \delta \frac{4\varepsilon^2 \partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{((e^{v/2})^2 + \varepsilon^2)^2} \\ &= \delta \frac{e^v}{(e^v + \varepsilon^2)} \left(\partial\bar{\partial}\log|T|^2 - \partial\bar{\partial}\lambda\right) + \delta \frac{4\varepsilon^2 \partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{((e^{v/2})^2 + \varepsilon^2)^2} \\ &= -\frac{\delta e^v}{(e^v + \varepsilon^2)} \partial\bar{\partial}\lambda + \delta \frac{4\varepsilon^2 \partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{((e^{v/2})^2 + \varepsilon^2)^2}. \end{aligned}$$

The last equality holds because $\partial\bar{\partial}\log|T|^2$ is supported on Z, where e^v vanishes.

We fix a $\mathscr{C}^2$-smooth function $h : [0,\infty) \to (0,\infty)$ to be determined momentarily, and set

$$\tau = (a + h(a)) \quad \text{and} \quad A = \frac{(1+h'(a))^2}{-h''(a)}.$$

With these choices, one calculates that

$$-\partial\bar{\partial}\tau - \frac{\partial\tau\wedge\bar{\partial}\tau}{A} = -(1+h'(a))\partial\bar{\partial}a - h''(a)\partial a\wedge\bar{\partial}a - \frac{(1+h'(a))^2}{A}\partial a\wedge\bar{\partial}a = (1+h'(a))(-\partial\bar{\partial}a).$$

Since the functions τ and A must be positive, we need

$$h'(x) > -1 \quad \text{and} \quad h''(x) < 0.$$

We shall need more from this function, though precisely what we need is perhaps unclear at present. We therefore make what looks like a rather inexplicable choice, setting

$$h(x) = 2 - x + \log(2e^{x-1} - 1).$$

4.5.2 Remark. There are some reasons for this choice, a few of which will become more clear as the proof progresses. There are, however, other possible choices for h. ⋄

We compute that

$$1 + h'(x) = \frac{2e^{x-1}}{2e^{x-1}-1} \quad \text{and} \quad -h''(x) = \frac{2e^{x-1}}{(2e^{x-1}-1)^2}$$

Finally, we take[1]

$$\psi_\nu = \varphi + \log(|T|^2e^{-\zeta} + \nu^2) + \zeta,$$

where $e^{-\zeta}$ is a positively curved metric for L_Z such that

$$\sup_X |T|^2e^{-\zeta} < +\infty.$$

For example, one could take $\zeta = \lambda + \rho$ for some bounded and sufficiently strictly plurisubharmonic function ρ; we have such a ρ because X is Stein and Ω is relatively compact. As in the computation of $-\partial\bar{\partial}a$, setting $w := \log|T|^2e^{-\zeta}$, we have

$$\begin{aligned}\partial\bar{\partial}\psi_\nu &= \partial\bar{\partial}\varphi + \partial\bar{\partial}\zeta - \frac{e^w}{e^w+\nu^2}\partial\bar{\partial}\zeta + 4\nu^2\frac{\partial(e^{w/2})\wedge\bar{\partial}(e^{w/2})}{(e^w+\nu^2)^2}\\ &= \partial\bar{\partial}\varphi + \frac{\nu^2}{e^w+\nu^2}\partial\bar{\partial}\zeta + 4\nu^2\frac{\partial(e^{w/2})\wedge\bar{\partial}(e^{w/2})}{(e^w+\nu^2)^2} \geq \partial\bar{\partial}\varphi \geq 0\end{aligned}$$

[1]We would like to take $\psi = \varphi + \log|T|^2$, but the latter is not smooth, so we regularize it in the obvious way.

and hence

$$\partial\bar{\partial}\psi_\nu - \delta\partial\bar{\partial}\lambda \geq \partial\bar{\partial}\varphi - \delta\partial\bar{\partial}\lambda \geq 0.$$

Then we have

$$\tau\partial\bar{\partial}\psi_\nu - \partial\bar{\partial}\tau - \frac{\partial\tau \wedge \bar{\partial}\tau}{A} = (a+h(a))\delta\frac{4\varepsilon^2\partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{((e^{v/2})^2+\varepsilon^2)^2}$$
$$+ \left(\tau - \frac{e^v(1+h'(a))}{e^v+\varepsilon^2}\right)\partial\bar{\partial}\psi_\nu + \frac{e^v(1+h'(a))}{e^v+\varepsilon^2}\left(\partial\bar{\partial}\psi_\nu - \delta\partial\bar{\partial}\lambda\right).$$

Now,

$$\tau - \frac{e^v(1+h'(a))}{e^v+\varepsilon^2} \geq 2 + \log(2e^{a-1}-1) - \frac{2e^{a-1}}{2e^{a-1}-1} > 0,$$

since the term between the two inequalities is an increasing function of a and $a > 1$. Moreover,

$$a + h(a) \geq a > 1,$$

and therefore

$$\tau\partial\bar{\partial}\psi_\nu - \partial\bar{\partial}\tau - \frac{\partial\tau \wedge \bar{\partial}\tau}{A} \geq \delta\frac{4\varepsilon^2\partial(e^{v/2}) \wedge \bar{\partial}(e^{v/2})}{((e^{v/2})^2+\varepsilon^2)^2}$$

Substitution into Lemma 4.5.1 then yields the following *a priori* estimate:

$$\delta\int_\Omega \frac{4\varepsilon^2\left|\left\langle\partial(e^{v/2}), u\right\rangle\right|^2}{((e^{v/2})^2+\varepsilon^2)^2}e^{-\psi_\nu} \leq ||\mathscr{D}^*u||^2_{\psi_\nu} + ||Su||^2_{\psi_\nu}, \tag{4.12}$$

where

$$\mathscr{D}\beta = \bar{\partial}\left(\sqrt{\tau+A}\beta\right) \quad \text{and} \quad Su = \sqrt{\tau}\left(\bar{\partial}u\right)$$

are two 'twisted $\bar{\partial}$ operators'. Note that $S\mathscr{D} = 0$.

4.5.3 A smooth extension and its holomorphic correction

From here on, let us fix the section $f \in H^0(Z, \mathcal{O}_Z(K_Z \otimes L)) \cap L^2(e^{-\varphi})$ to be extended.

Since X is Stein, Theorem 4.2.1 yields a section $\tilde{f} \in H^0(X, \mathcal{O}_X(K_X \otimes L_Z \otimes L))$ such that

$$\tilde{f}|_Z = f \wedge dT.$$

If we take any relatively compact domain $\Omega \subset\subset X$ then the smoothness of the metrics immediately implies that

$$\int_\Omega |\tilde{f}|^2 e^{-(\varphi+\lambda)} < +\infty.$$

However, we do not have any control over this quantity as we increase Ω to exhaust X. In order to tame the growth of this extension $\tilde{f}$, we first modify it to a smooth extension, and then correct this smooth extension to be holomorphic by solving a twisted version of the $\bar{\partial}$-equation.

Turning to the smooth modification, let $t \in (0, 1/2)$ and fix $\chi \in \mathscr{C}^\infty_o([0,1))$ with

$$0 \le \chi \le 1, \quad \chi \equiv 1 \text{ on } [0,t] \quad \text{and} \quad |\chi'| \le 1 + 2t.$$

We write

$$\chi_\varepsilon := \chi\left(\frac{|T|^2 e^{-\lambda}}{\varepsilon^2}\right).$$

We distinguish the $(n,1)$-form

$$\alpha_\varepsilon := \bar\partial \chi_\varepsilon \tilde f = \tilde f \wedge \chi'(\varepsilon^{-2}e^v)\varepsilon^{-2}\bar\partial e^v = \chi'(\varepsilon^{-2}e^v)\frac{2e^{v/2}}{\varepsilon^2}\tilde f \wedge \bar\partial(e^{v/2}).$$

Then one has the estimate

$$\begin{aligned}
\left|(u,\alpha_\varepsilon)_{\psi_\nu}\right|^2 &\le \left(\int_\Omega |\langle u,\alpha_\varepsilon\rangle| e^{-\psi_\nu}\right)^2 \\
&= \left(\int_\Omega \left|\left\langle u, \chi'\left(\frac{e^v}{\varepsilon^2}\right)\tilde f \wedge \frac{2e^{v/2}\bar\partial(e^{v/2})}{\varepsilon^2}\right\rangle\right| e^{-\psi_\nu}\right)^2 \\
&\le \frac{1}{\delta}\int_\Omega \left|\frac{\tilde f}{\varepsilon^2}\chi'\left(\frac{e^v}{\varepsilon^2}\right)\right|^2 \frac{(e^v+\varepsilon^2)^2}{\varepsilon^2} e^{-(\varphi+\lambda)} \\
&\quad \times \delta \int_\Omega \left|\left\langle u, \bar\partial(e^{v/2})\right\rangle\right|^2 \frac{4\varepsilon^2}{(e^v+\varepsilon^2)^2} e^{-\psi_\nu} \\
&\le C_\varepsilon\left(||\mathscr{D}^* u||^2_{\psi_\nu} + ||Su||^2_{\psi_\nu}\right)
\end{aligned}$$

where

$$C_\varepsilon := \frac{4(1+t)^2}{\delta\varepsilon^2}\int_{e^v \le \varepsilon^2} |\tilde f|^2 e^{-(\varphi+\lambda)}.$$

The last inequality follows from the a priori estimate (4.12) and the fact that

(i) $|\chi'| \le 1 + t$, and

(ii) $t\varepsilon^2 \le e^v \le \varepsilon^2$ on the support of $\chi'(e^v/\varepsilon^2)$.

We note for later use that

$$\limsup_{\varepsilon\to 0} C_\varepsilon = \frac{8\pi(1+t)^2}{\delta}\int_Z |f|^2 e^{-\varphi}.$$

By the usual L^2-method, followed by letting $\nu \to 0$ via weak-$*$ compactness arguments we have used many times before, we obtain the following result.

4.5.3 Theorem. *Let*

$$\psi = \psi_0 = \varphi + \log |T|^2.$$

Then there exists a smooth section $\beta_\varepsilon \in \Gamma(X, K_X \otimes L_Z \otimes L)$ *such that*

$$\mathscr{D}\beta_\varepsilon = \alpha_\varepsilon \quad \text{and} \quad \int_\Omega |\beta_\varepsilon|^2 e^{-\psi} \le C_\varepsilon.$$

In particular, since $e^{-\psi} = \frac{e^{-\varphi}}{|T|^2}$,

$$\beta_\varepsilon|_Z \equiv 0.$$

4.5.4 End of the proof of Theorem 4.3.1

We begin by observing that, since

$$e^v(\tau + A) = \left(e^{\delta^{-1}(\gamma - a)} - \varepsilon^2\right)(2 + \log(2e^{a-1} - 1) + 2e^{a-1})$$

and $\delta \le 1$, the inequality $2 + \log(x-1) \le x$ for $x > 1$, which one can prove by using elementary calculus to show that the maximum value 0 of the function $\phi(x) = 2 - x + \log(x-1)$ is achieved at $x = 2$, shows that

$$\sup_X e^v(\tau + A) \le \sup_{a \ge \gamma} e^{\gamma - a}(2 + \log(2e^{a-1} - 1) + 2e^{a-1}) \le \sup_{a \ge \gamma} X e^{\gamma - a}(2 + 4e^{a-1}) \le 6e^{\gamma - 1}. \tag{4.13}$$

where we have used $\log(x-1) \le x - 2$ for $x > 1$.

Let

$$F_\varepsilon = \left(\chi_\varepsilon \tilde{f} - \sqrt{\tau + A}\beta_\varepsilon\right).$$

Note that by Theorem 4.5.3 $F_\varepsilon|_Z = f$ and that by Theorem 4.5.3 and (4.13)

$$\int_\Omega |F_\varepsilon|^2 e^{-(\varphi + \lambda)} \le o(1) + \int_\Omega e^v(\tau + A)|\beta_\varepsilon|^2 e^{-\psi} \le o(1) + \frac{48\pi(1+t)^2 e^{\gamma - 1}}{\delta} \int_Z |f|^2 e^{-\varphi}$$

for $\varepsilon \sim 0$. Indeed, the term $\chi_\varepsilon \tilde{f}$ is supported on a set whose measure converges to 0 with ε.

Using the already-familiar technique of Alaoglu's Theorem and dominated convergence, we can let $\varepsilon \to 0$ and $t \to 0$. We can let the smooth metrics converge to singular metrics. And finally we can let $\Omega \nearrow X$. Details, which have been presented a number of times by now, are left to the reader.

Finally, since L^2 convergence implies pointwise convergence for holomorphic sections, we see that also $F|_Z = f \wedge dT$. Thus we obtain a section $F \in H^0(X, K_X \otimes L_Z \otimes L)$ satisfying

$$F|_Z = f \wedge dT \quad \text{and} \quad \int_X |F|^2 e^{-(\varphi + \lambda)} \le \frac{48\pi}{\delta} \int_Z |f|^2 e^{-\varphi}.$$

The proof of Theorem 4.3.1 is complete. □